

# Fourier Transform I

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## Application of Fourier Transform (FT)

- **Evaluation of image quality**

- MTF: resolution property

- Wiener spectrum: noise property

- **Image processing in frequency domain**

- Low pass filter

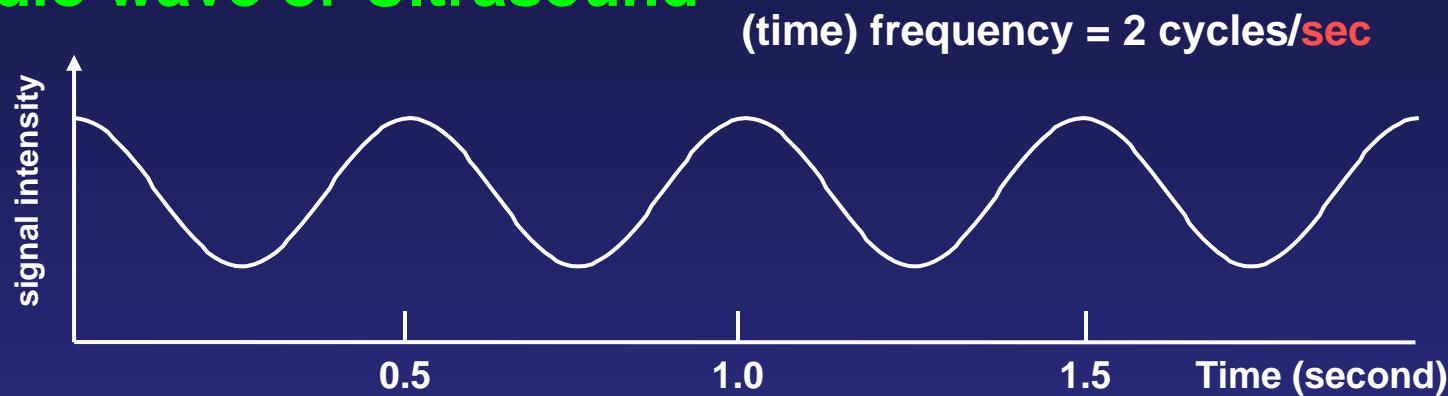
- **Image reconstruction of MR image**

- MR signal is obtained in frequency domain (k-space).

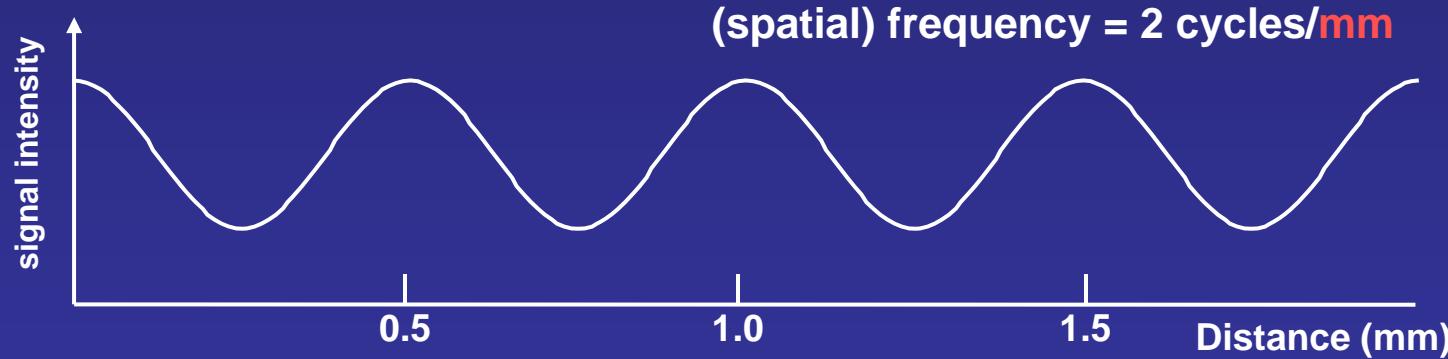
- MRI in real domain is obtained by FT.

## Time Frequency vs. Spatial Frequency

Radio wave or Ultrasound

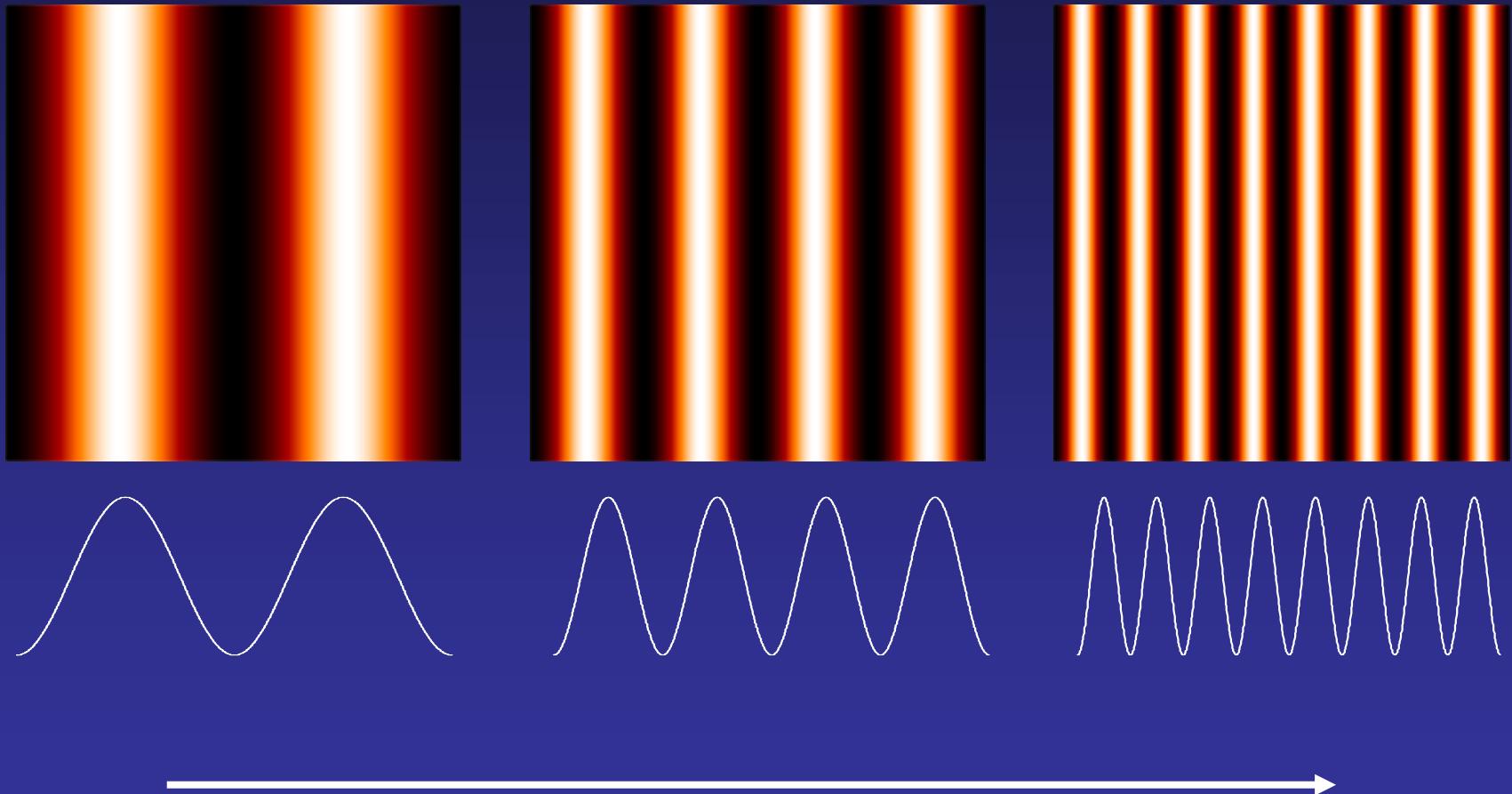


Brightness variation in an image



FT can be applied for both time and spatial frequencies.

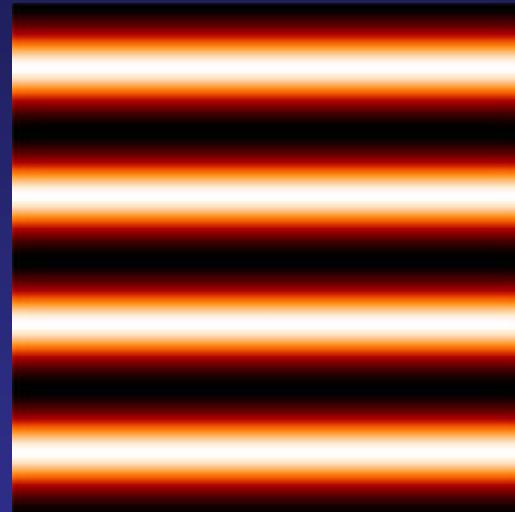
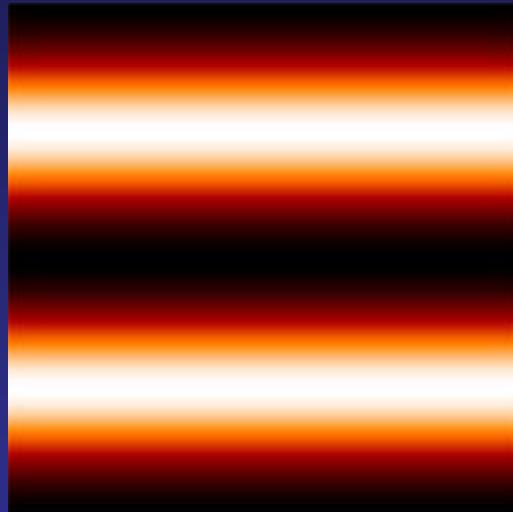
## Wave propagating in Horizontal Direction



Low spatial freq.

High spatial freq.

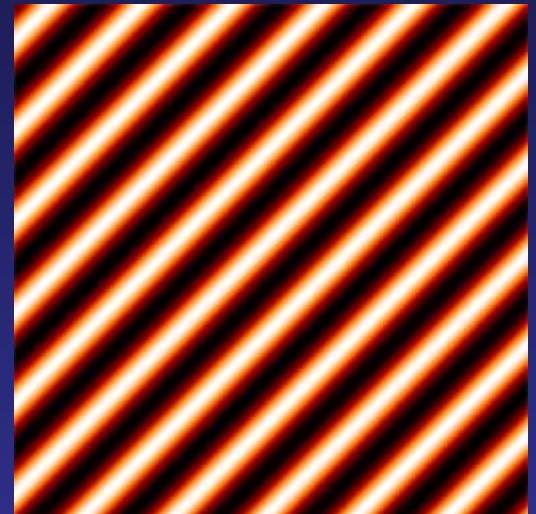
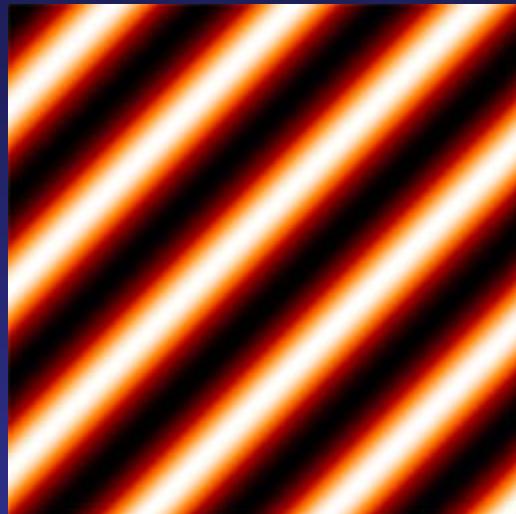
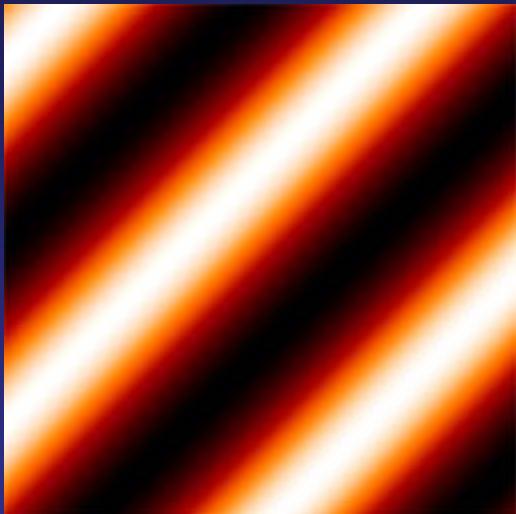
## Wave propagating in Vertical Direction



Low spatial freq.

High spatial freq.

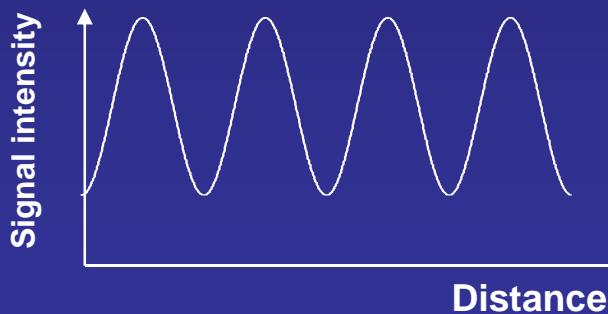
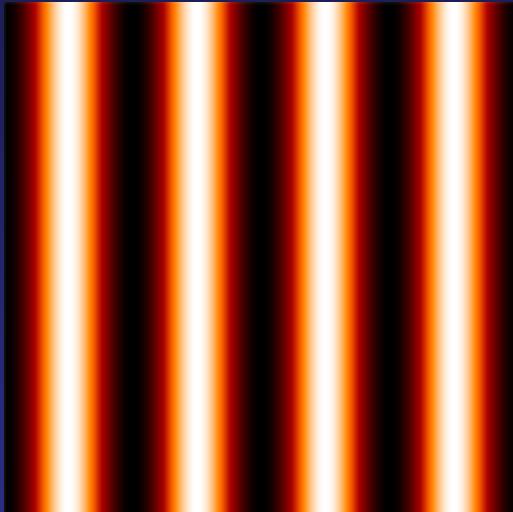
## Wave propagating in Diagonal Direction



Low spatial freq.

High spatial freq.

## Wave propagating in Real Domain



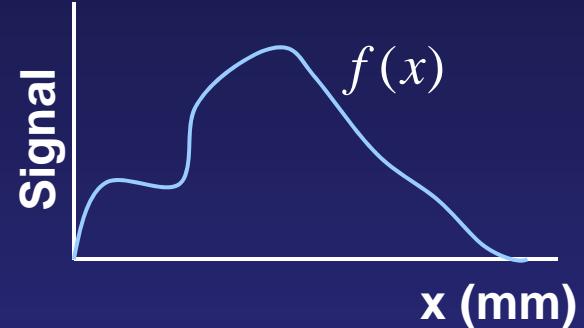
**Signal in an image:**  
**Optical density on radiograph**  
**Brightness on display monitor**

**Distance (unit of spatial freq.):**  
**mm (cycles/mm)**  
**cm (cycles/cm)**

# Function in Mathematical Expression

**Function  $f(x)$  :**

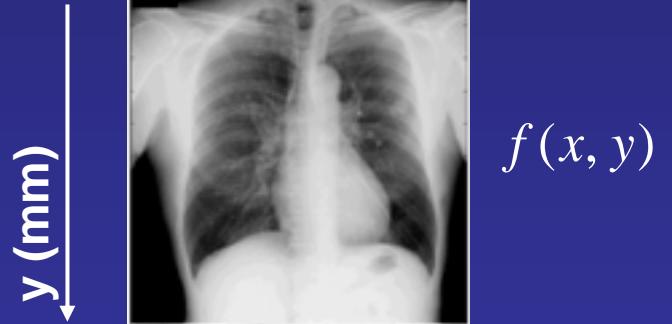
Signal variation along x-axis (distance)



**Image  $f(x, y)$**

2D function

Signal variation along x- and y-axes



We explain FT by using a 1D function for simplicity.

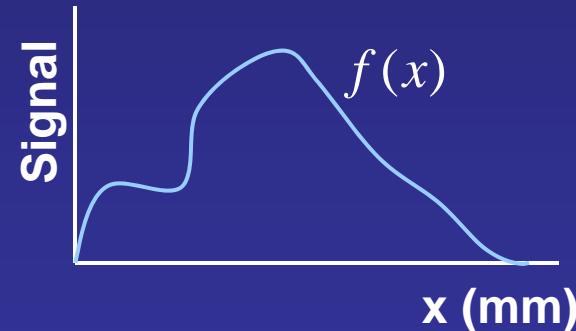
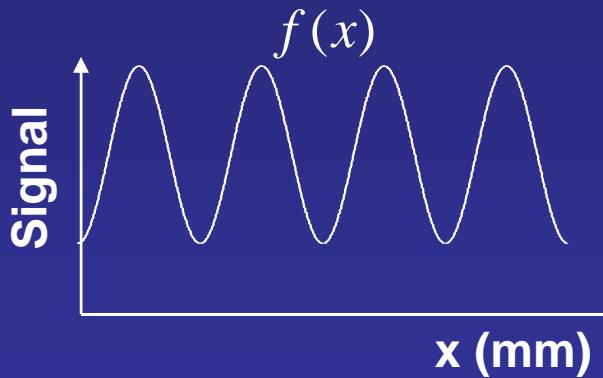
## Expression of Function in Spatial Freq. Domain

Real Domain

Spatial Freq. Domain

Periodic function  $\rightarrow$  Fourier Series

Non-periodic function  $\rightarrow$  Fourier Transform

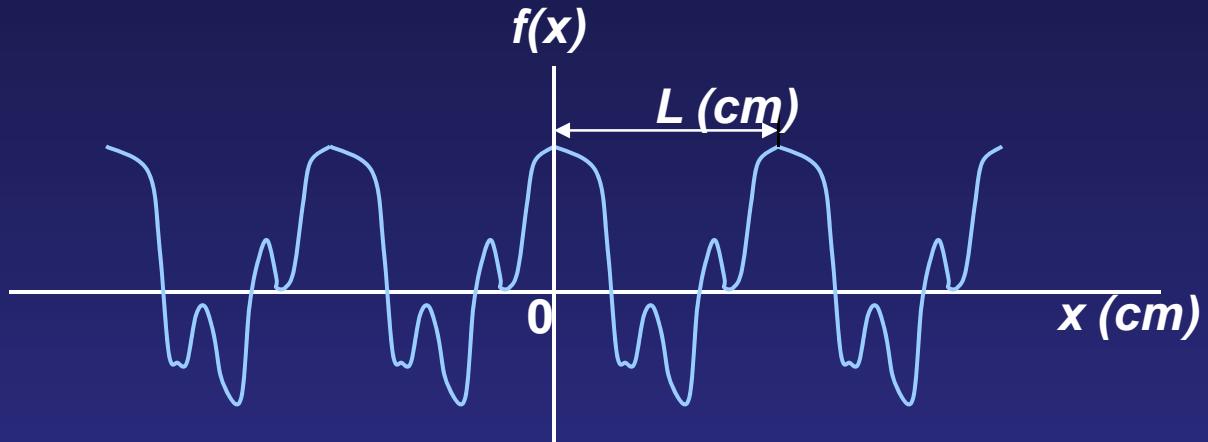


Medical images are 2D non-periodic functions.

**Fourier transform is derived from Fourier series.**

**Therefore, we have to start from learning a periodic function in real domain and corresponding Fourier series in frequency domain.**

# Periodic Function



The signal is repeating with  $L$  (cm) period.

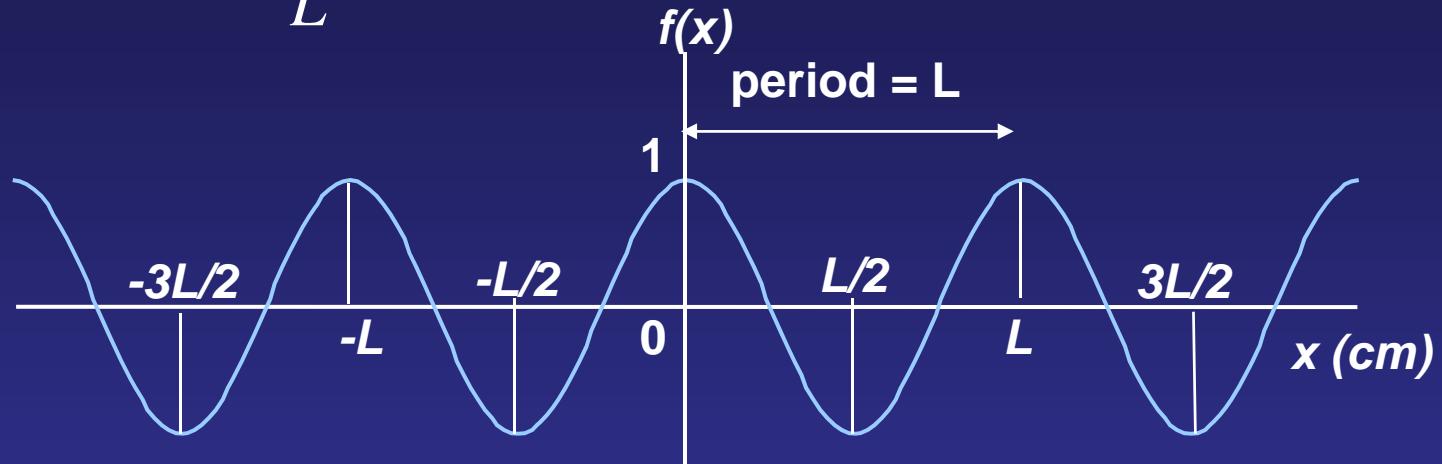
$$f(x) = f(x + nL)$$

$$n = 1, 2, 3, \dots$$

$L$ : period

## Cosine Function as Simplest Periodic Function

$$f(x) = \cos\left(\frac{2\pi}{L}x\right)$$

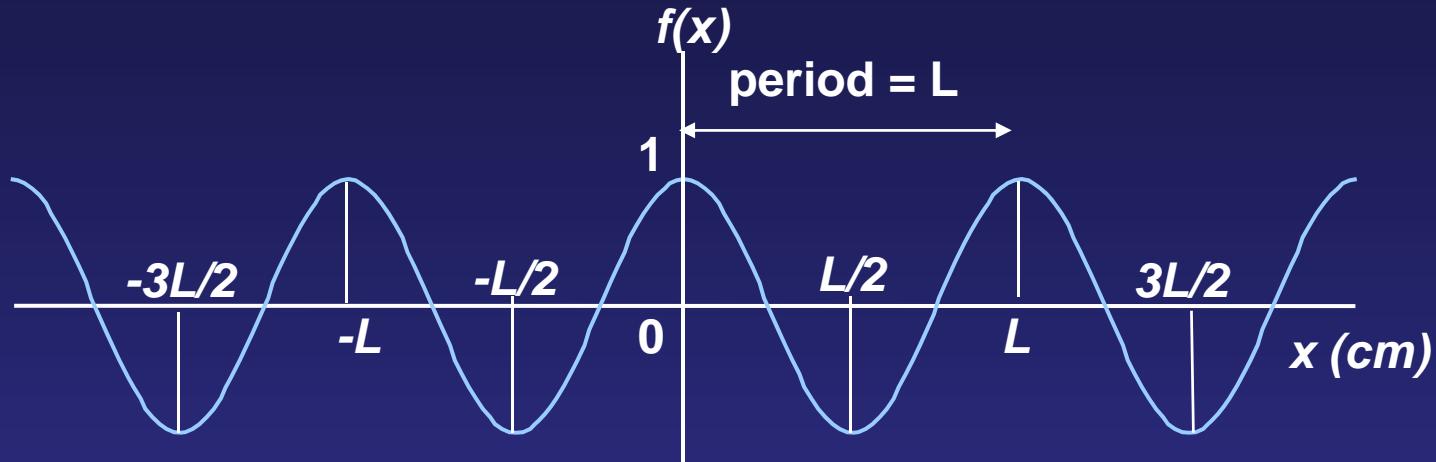


$$f(x + L) = \cos\left\{\frac{2\pi}{L}(x + L)\right\} = \cos\left(\frac{2\pi}{L}x + 2\pi\right) = \cos\left(\frac{2\pi}{L}x\right) = f(x)$$

$$f(x + 2L) = \cos\left\{\frac{2\pi}{L}(x + 2L)\right\} = \cos\left(\frac{2\pi}{L}x + 4\pi\right) = \cos\left(\frac{2\pi}{L}x\right) = f(x)$$

.....

## Period vs. Spatial Frequency



$$f(x) = \cos\left(\frac{2\pi}{L}x\right) = \cos(2\pi ux)$$

$u = 1/L$        $u$  (cycles/cm): spatial frequency

$$f(x) = \cos\left(\frac{2\pi}{L}x\right) = \cos(2\pi ux) = \cos(\omega_0 x)$$

$\omega_0 = \frac{2\pi}{L} = 2\pi u$        $\omega_0$  (radian/cm): angular frequency

# Fourier Series Expansion of Periodic Function

## Mathematical Fact

Any periodic function can be obtained by summation of trigonometric (cosine and sine) function.

If  $f(x)$  is periodic function with period  $L$ ,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)\} \quad (2.6)$$

where  $\omega_0 = 2\pi / L$

$$a_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos(n\omega_0 x) dx \quad n = 0, 1, 2, 3, \dots \quad (2.7)$$

$$b_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin(n\omega_0 x) dx \quad n = 0, 1, 2, 3, \dots \quad (2.8)$$

$$\frac{a_0}{2} = \frac{1}{L} \int_{-L/2}^{L/2} f(x) dx \quad \text{average signal intensity (DC component)}$$

$\cos(\omega_0 x), \sin(\omega_0 x) \dots$  fundamental harmonic

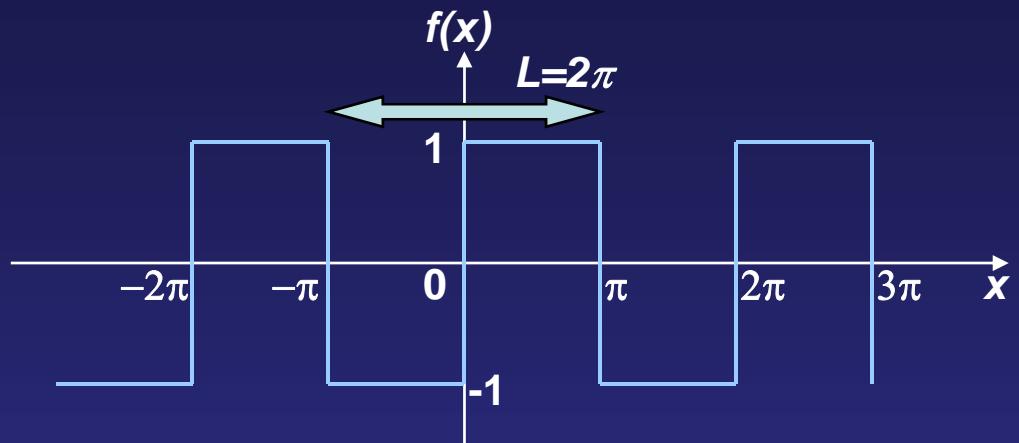
$\cos(n\omega_0 x), \sin(n\omega_0 x) \dots$   $n^{th}$  harmonic

## Example: Fourier Series Expansion 1

**Square Wave**

$$f(x) = \begin{cases} -1 & (-\pi \leq x < 0) \\ 1 & (0 \leq x < \pi) \end{cases}$$

$$L = 2\pi \Rightarrow \omega_0 = 2\pi / L = 1$$



$$a_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos(n\omega_0 x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = 0$$

$$b_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin(n\omega_0 x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} \sin(nx) dx$$

$$= \begin{cases} \frac{4}{n\pi} & (n: \text{odd number}) \\ 0 & (n: \text{even number}) \end{cases}$$

(Note) Consider integral of the odd and even function!

## Example: Fourier Series Expansion 2

$$\omega_0 = 1, \quad a_n = 0, \quad b_n = b_{2k+1} = \frac{4}{(2k+1)\pi}$$

Therefore,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)\} = \sum_{k=0}^{\infty} b_{2k+1} \sin((2k+1)x)$$

Approximation,

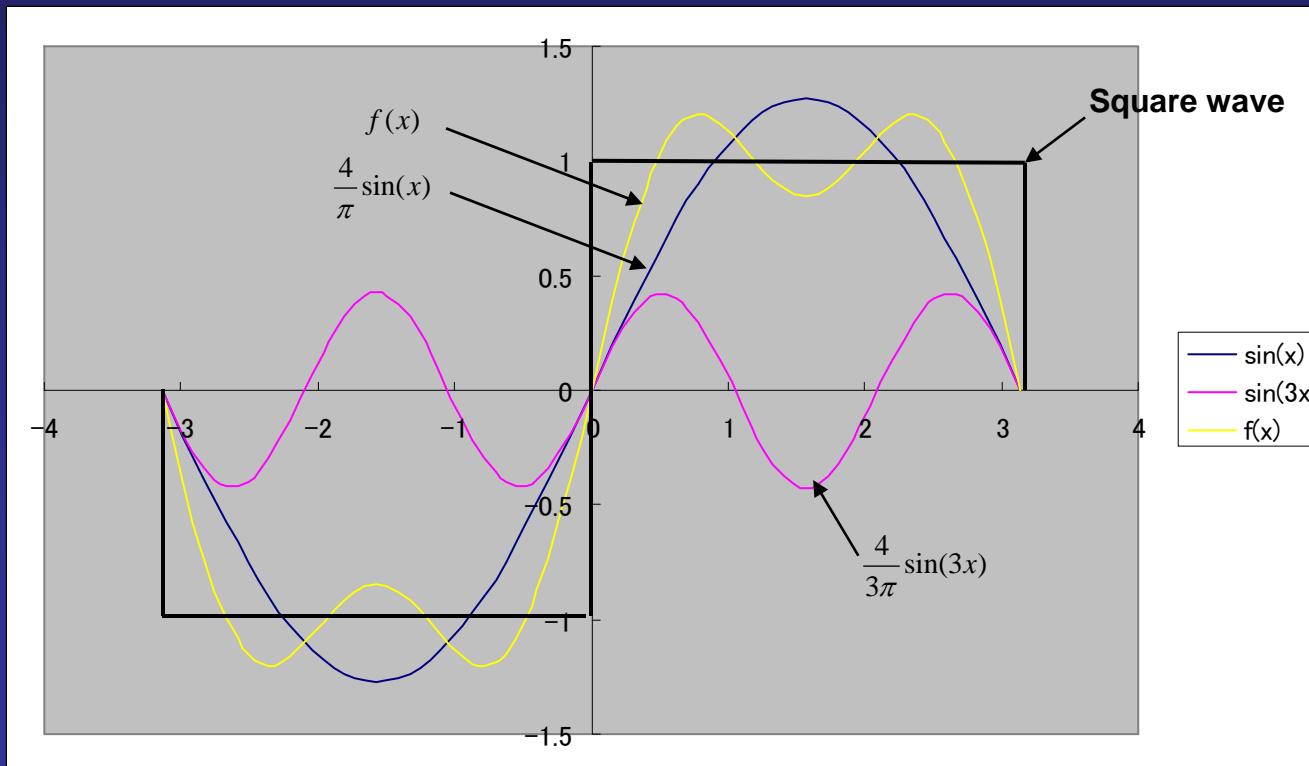
$$f(x) \approx \frac{4}{\pi} \sum_{k=0}^K \frac{1}{2k+1} \sin((2k+1)x)$$

A square wave can be expressed by summation of sine functions.

## Example: Fourier Series Expansion 3

Approximation with first two terms,

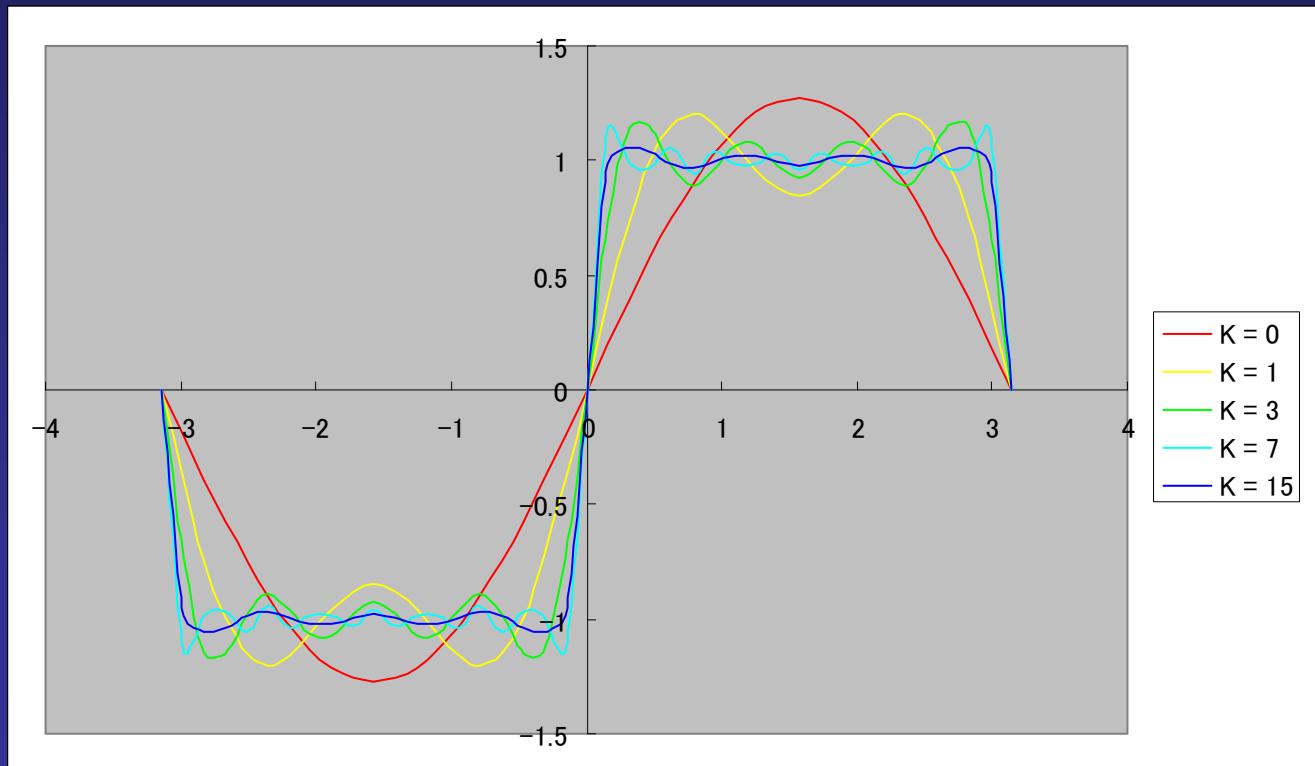
$$f(x) \approx \frac{4}{\pi} \left\{ \sin(x) + \frac{1}{3} \sin(3x) \right\}$$



## Example: Fourier Series Expansion 4

$$f(x) \approx \frac{4}{\pi} \sum_{k=0}^K \frac{1}{2k+1} \sin((2k+1)x)$$

Approximation with 1-16 (K=0-15) terms,



# Complex Number

The complex number makes it easier to express Fourier transform.

**z: complex number**

**a: real part**

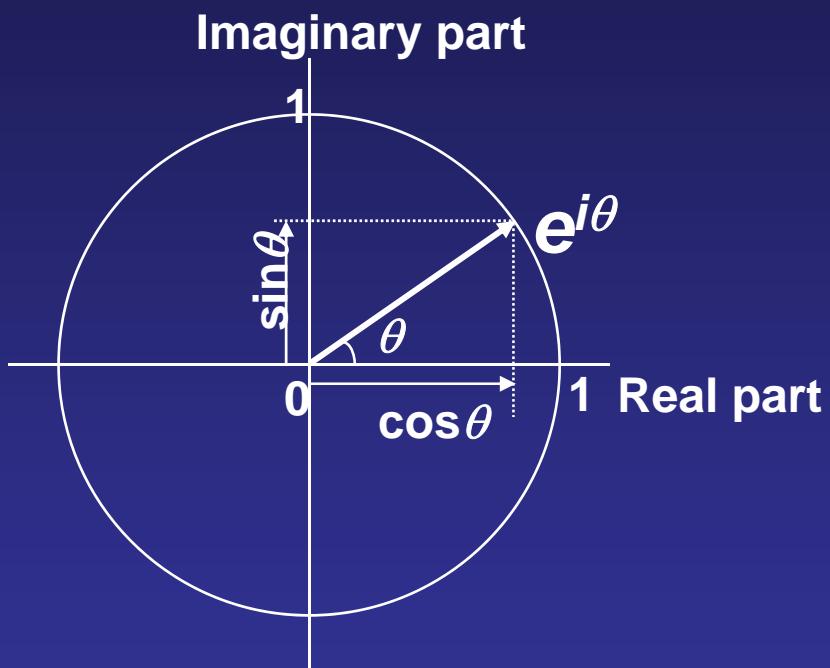
**$z = a + ib$**       **b: imaginary part**

**i: imaginary unit,  $i^2 = -1$**

**Complex conjugate number of z,**

$$z^* = a - ib$$

## Complex Exponential in Unit Circle (Euler's Eq.)



$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

## Complex Fourier Series 1

From eq. (2.6),

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)\} \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \frac{e^{in\omega_0 x} + e^{-in\omega_0 x}}{2} + b_n \frac{e^{in\omega_0 x} - e^{-in\omega_0 x}}{2i} \right\} \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ \frac{1}{2}(a_n - ib_n)e^{in\omega_0 x} + \frac{1}{2}(a_n + ib_n)e^{-in\omega_0 x} \right\} \end{aligned} \quad (2.15)$$

The complex Fourier Coefficient,  $c_n$ , is defined as follows,

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2} \quad (2.16)$$

From eq. (2.15),  $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}$ ; complex Fourier series  
(2.19)

## Complex Fourier Series 2

From eq. (2.15), the **complex Fourier coefficient** is obtained as follows;

$$\begin{aligned} c_n &= \frac{a_n - ib_n}{2} \\ &= \frac{1}{2} \left\{ \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos(n\omega_0 x) dx - i \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin(n\omega_0 x) dx \right\} \\ &= \frac{1}{L} \int_{-L/2}^{L/2} f(x) \{ \cos(n\omega_0 x) - i \sin(n\omega_0 x) \} dx \\ &= \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-n\omega_0 x} dx \end{aligned} \tag{2.20}$$

The complex Fourier series and coefficient are simple equations in comparison with those including trigonometric functions.

## Complex Fourier Series 3

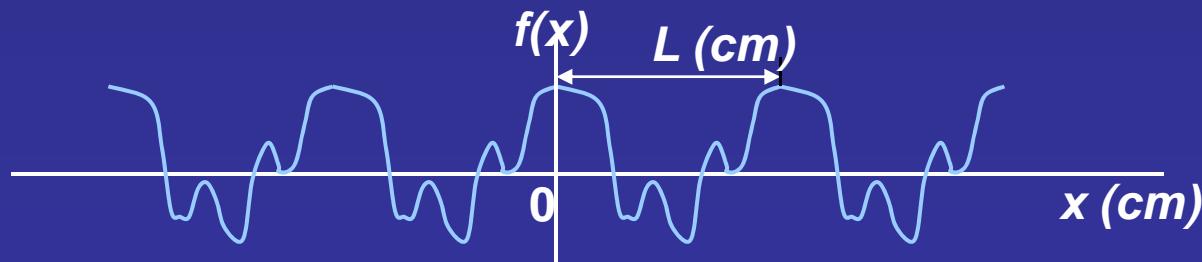
In summary, the **complex Fourier series** of  $f(x)$  is defined as follows;

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} \quad (2.19)$$

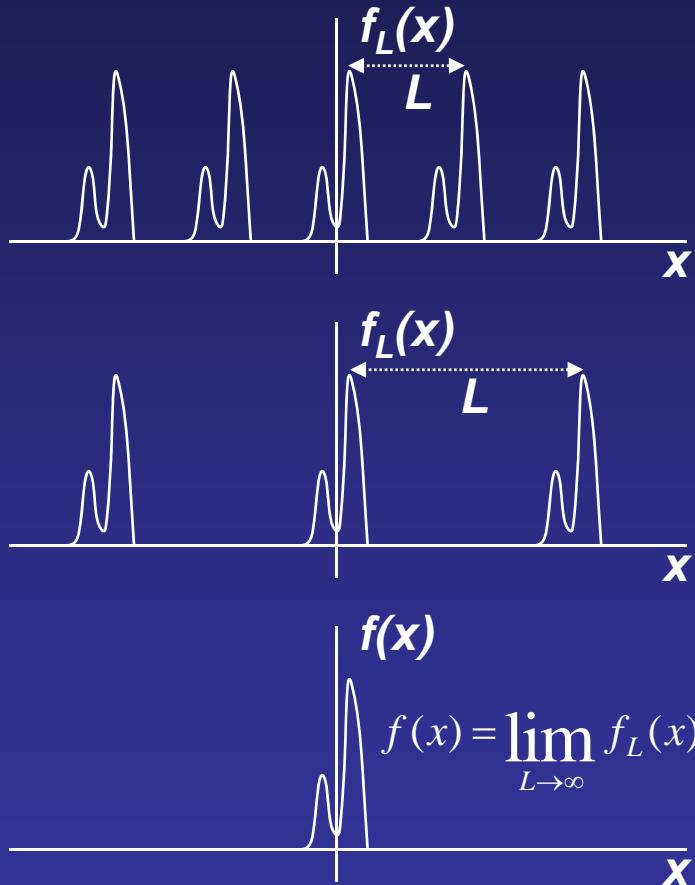
The **complex Fourier coefficient**,  $c_n$ , is obtained as follows;

$$c_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-n\omega_0 x} dx \quad (2.20)$$

where  $\omega_0 = \frac{2\pi}{L}$ ,  $L$  is a period of periodic function,  $f(x)$ .



# Fourier Transform 1



$f_L(x)$  : periodic function with finite  $L$

General functions such as image are non-periodic functions,  $f(x)$ .

Non-periodic functions are considered as periodic functions with infinite  $L$ .

Therefore, Fourier transform (FT) is obtained from Fourier series with infinite  $L$ .

## Fourier Transform 2

**From (2.19) and (2.20),**  $f_L(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}, \quad c_n = \frac{1}{L} \int_{-L/2}^{L/2} f_L(x) e^{-n\omega_0 x} dx$

$$\omega_0 = \frac{2\pi}{L} ; \quad \frac{1}{L} = \frac{\omega_0}{2\pi}, \quad f_L(x) = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-L/2}^{L/2} f_L(\xi) e^{-in\omega_0 \xi} d\xi \right] \omega_0 e^{in\omega_0 x}$$

$$L \rightarrow \infty, \quad f_L(x) \rightarrow f(x), \quad \omega_0 \rightarrow d\omega, \quad n\omega_0 \rightarrow \omega, \quad \sum_{n=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty}$$

$$f(x) = \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega \xi} d\xi \right] e^{i\omega x} d\omega$$

**However, the following relationship is required,**

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty$$

## Fourier Transform 3

$$f(x) = \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi \right] e^{i\omega x} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \quad \text{Fourier Transform} \quad (2.24)$$

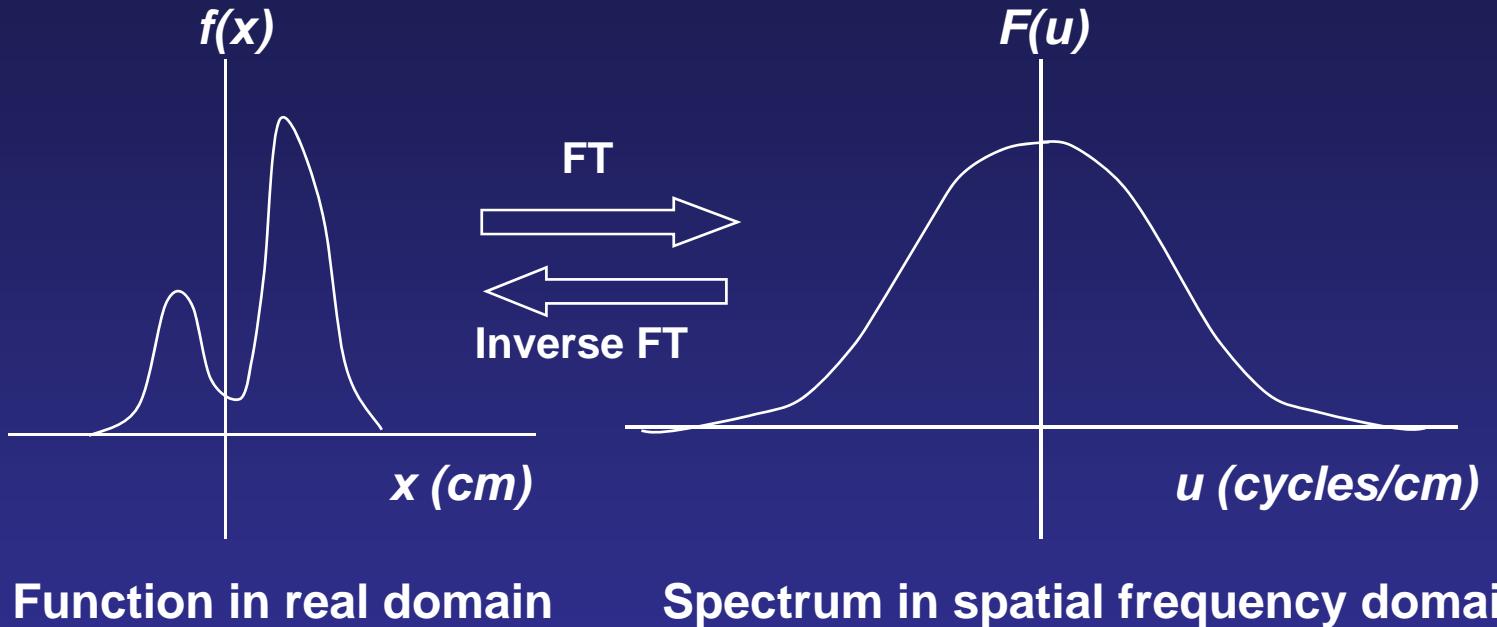
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega \quad \text{Inverse Fourier Transform} \quad (2.25)$$

$\omega = 2\pi u, \quad d\omega = 2\pi du, \quad \omega: \text{angular freq.}, \quad u: \text{spatial freq.}$

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} du$$

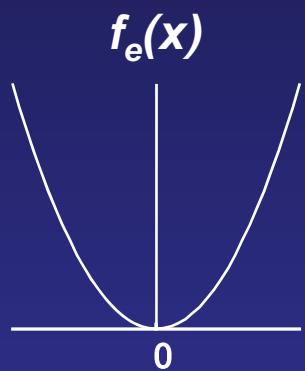
## Fourier Transform 4



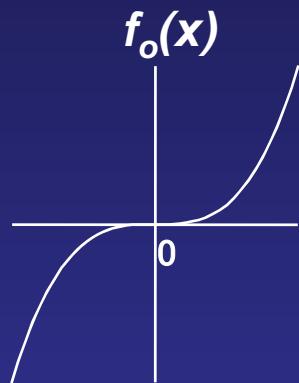
Function,  $f(x)$ , includes the distribution of spatial frequency components (spectrum) as shown by  $F(u)$ .

# Even/Odd Function

Even function,  $f_e(x)$



Odd function,  $f_o(x)$



$$f_e(-x) = f_e(x)$$

$$\int_{-\infty}^{\infty} f_e(x) dx = 2 \int_0^{\infty} f_e(x) dx$$

$$\cos(-x) = \cos(x)$$

$$f_o(-x) = -f_o(x)$$

$$\int_{-\infty}^{\infty} f_o(x) dx = 0$$

$$\sin(-x) = -\sin(x)$$

$$f_e(-x)g_e(-x) = f_e(x)g_e(x); \text{ even function}$$

$$f_e(-x)g_o(-x) = -f_e(x)g_o(x); \text{ odd function}$$

$$f_o(-x)g_e(-x) = -f_o(x)g_e(x); \text{ odd function}$$

$$f_o(-x)g_o(-x) = f_o(x)g_o(x); \text{ even function}$$

## Fourier Transform of Even/Odd Function

### FT of even function, $f_e(x)$

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f_e(x) e^{-i\omega x} dx \\ &= \int_{-\infty}^{\infty} f_e(x) \cos(\omega x) dx - i \int_{-\infty}^{\infty} f_e(x) \sin(\omega x) dx \\ &= 2 \int_0^{\infty} f_e(x) \cos(\omega x) dx \quad ; \text{real number} \end{aligned}$$

### FT of odd function, $f_o(x)$

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f_o(x) e^{-i\omega x} dx \\ &= \int_{-\infty}^{\infty} f_o(x) \cos(\omega x) dx - i \int_{-\infty}^{\infty} f_o(x) \sin(\omega x) dx \\ &= -2i \int_0^{\infty} f_o(x) \sin(\omega x) dx \quad ; \text{imaginary number} \end{aligned}$$

## Symmetric Property of Fourier Transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega \quad (2.25)$$

$$f(-x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$$

$$f(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) e^{-i\omega x} dx \quad ; x \leftrightarrow \omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \mathfrak{I}[f(x)] \quad : \mathfrak{I}[f(x)] \text{ means FT of } f(x).$$

$$\mathfrak{I}[F(x)] = f(-\omega) \quad \text{symmetric property of FT}$$

If  $F(\omega)$  is FT of  $f(x)$ , FT of  $F(x)$  is  $f(-\omega)$ .

## Parseval's Theorem

$$F(\omega)F^*(\omega) = |F(\omega)|^2, \quad F^*(\omega): \text{complex conjugate}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)F^*(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left[ \int_{-\infty}^{\infty} f(x)e^{i\omega x} dx \right] d\omega$$

$$= \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega x} d\omega \right] f(x) dx$$

$$= \int_{-\infty}^{\infty} f(x)^2 dx$$

$$\int_{-\infty}^{\infty} f(x)^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega ; \text{Parsevals Theorem} \quad (2.35)$$

**Total power in real domain is same as that in spatial frequency domain.**

$|F(\omega)|^2$  ; Power Spectrum

## Convolution Integral Theorem

The convolution integral of  $f(x)$  and  $g(x)$  is defined as follows,

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\tau)g(x - \tau)d\tau$$

The convolution integral has a strong relationship with Fourier transform.

If  $F(\omega) = \mathfrak{F}[f(x)]$  and  $G(\omega) = \mathfrak{F}[g(x)]$ ,

$$\begin{aligned}\mathfrak{F}[f(x) * g(x)] &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\tau)g(x - \tau)d\tau \right] e^{-i\omega x} dx \\ &= \int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau} d\tau \int_{-\infty}^{\infty} g(x - \tau)e^{-i\omega(x-\tau)} dx \quad ; e^{-iw\tau} e^{iw\tau} = e^0 = 1 \\ &= \int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau} d\tau \int_{-\infty}^{\infty} g(x)e^{-i\omega x} dx \quad ; x - \tau \rightarrow x \\ &= F(\omega)G(\omega)\end{aligned}$$

$$\mathfrak{F}[f(x) * g(x)] = F(\omega)G(\omega) \quad \text{convolution integral theorem (2.37)}$$

FT of convolution of  $f(x)$  and  $g(x)$  is equal to the multiplication of each FT.

**We will learn the application of Fourier transform in next G-Class.**