G-Class of Medical Image Information

Fourier Transform II

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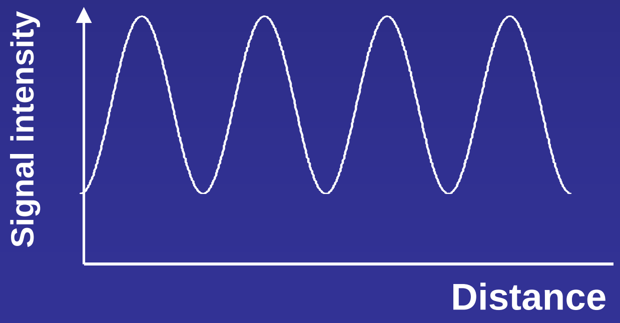
Wave propagating in Real Domain



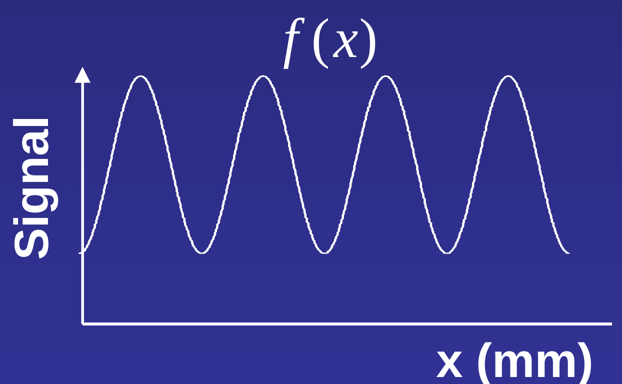
Signal in an image:
- Optical density on radiograph
- Brightness on display monitor

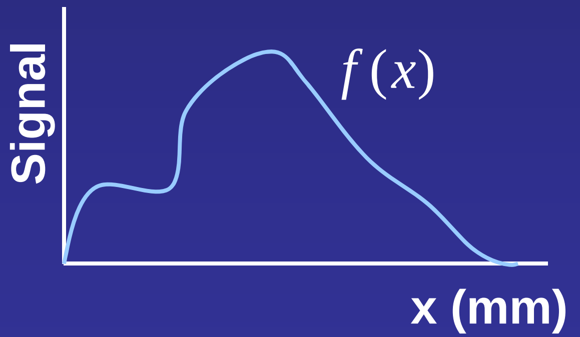
Spatial Frequency is defined as the number of waves in a unit length

Distance (unit of spatial freq.):
- mm (cycles/mm)
- cm (cycles/cm)

Expression of Function in Spatial Freq. Domain

Real Domain		Spatial Freq. Domain
Periodic function	→	Fourier Series
Non-periodic function	→	Fourier Transform





Medical images are 2D non-periodic functions.

Complex Fourier Series

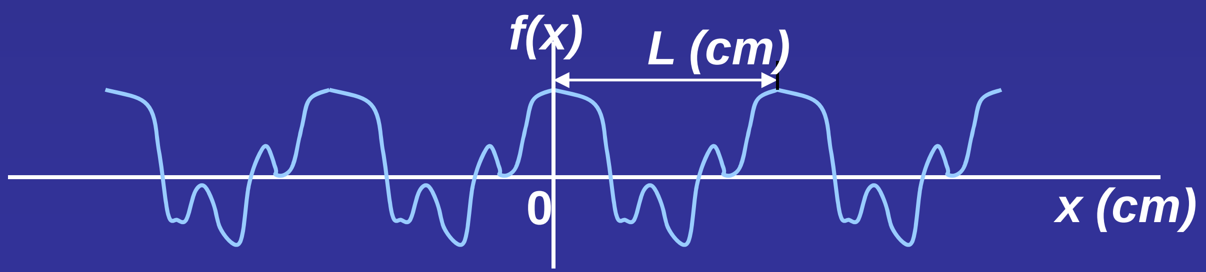
The complex Fourier series of f(x) is defined as follows;

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} \qquad (2.19)$$

The complex Fourier coefficient, c_n, is obtained as follows;

$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-in\omega_0 x} dx \qquad (2.20)$$

$$where \quad \omega_0 = \frac{2\pi}{L}, \quad L \; is \; a \; period \; of \; periodic \; function, \; f(x).$$



Fourier Transform

$$f(x) = \int_{-\infty}^{\infty} [\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi] e^{i\omega x} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

Fourier Transform (2.24)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

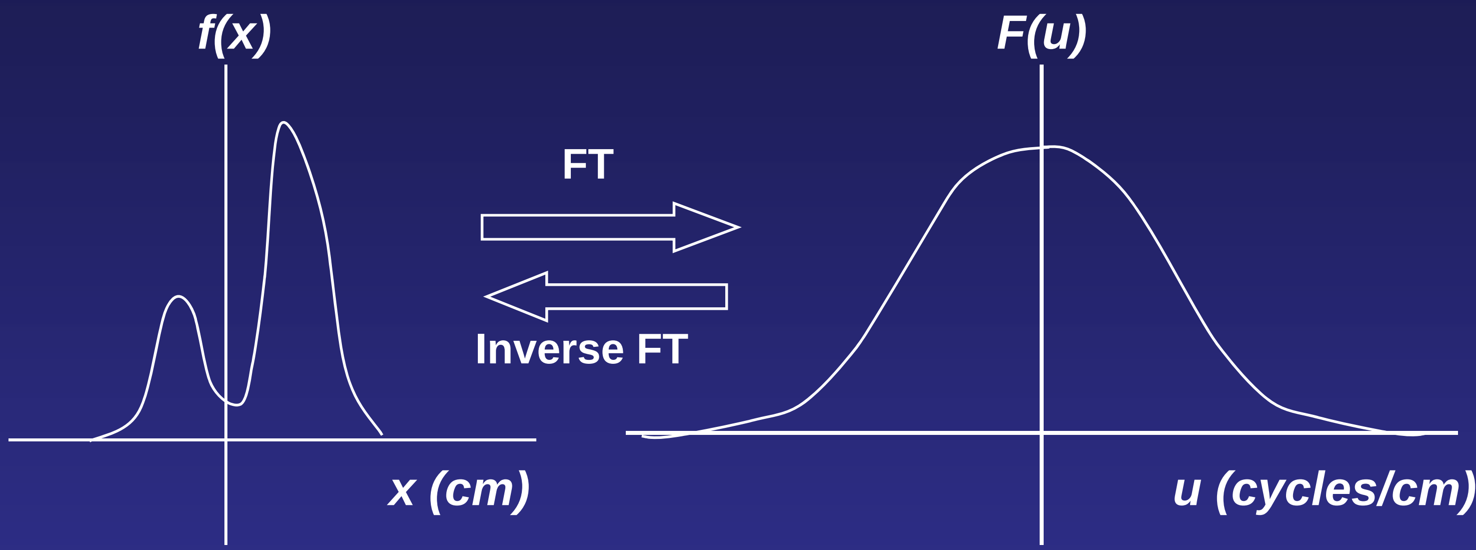
Inverse Fourier Transform (2.25)

$$\omega = 2\pi u, \quad d\omega = 2\pi du, \quad \omega : angular\ freq., \quad u : spatial\ freq.$$

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} du$$

Fourier Transform



Function in real domain

Spectrum in spatial frequency domain

Function, *f(x),* includes the distribution of spatial frequency components (spectrum) as shown by *F(u).*

Fourier Transform of Image

FT

Inverse FT

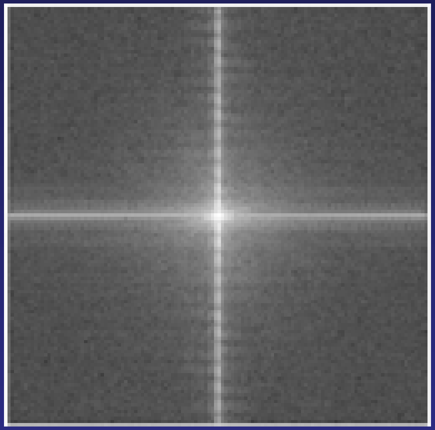


Image in real domain

Spectrum in spatial frequency domain

Parseval's Theorem and Convolution Integral

$$\int_{-\infty}^{\infty} f(x)^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|F(\omega)\right|^2 d\omega \quad ; Parseval's\ \ Theorem$$

$$\left|F(\omega)\right|^2 \quad ; Power\ Spectrum$$

The convolution integral of *f(x)* and *g(x)* is defined as follows,

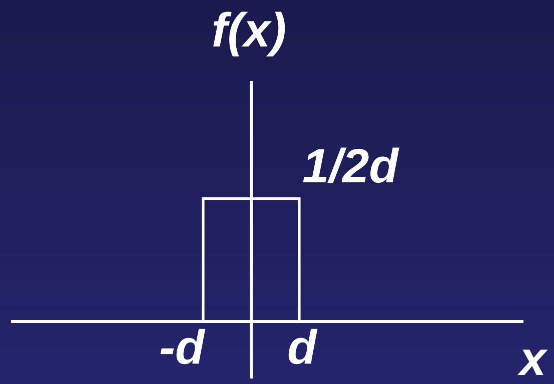
$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\tau) g(x-\tau) d\tau$$

$$If \quad F(\omega) = \Im[f(x)]\ and\ G(\omega) = \Im[g(x)]\,,$$

$$\Im[f(x) * g(x)] = F(\omega)G(\omega)$$ **convolution integral theorem**

FT of convolution of f(x) and g(x) is equal to the multiplication of each FT.

Fourier Transform of Rectangle Pulse 1



$$f(x) = \begin{cases} 1/2d & (|x| \le d) \\ 0 & (|x| > d) \end{cases}$$

From (2.24),

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

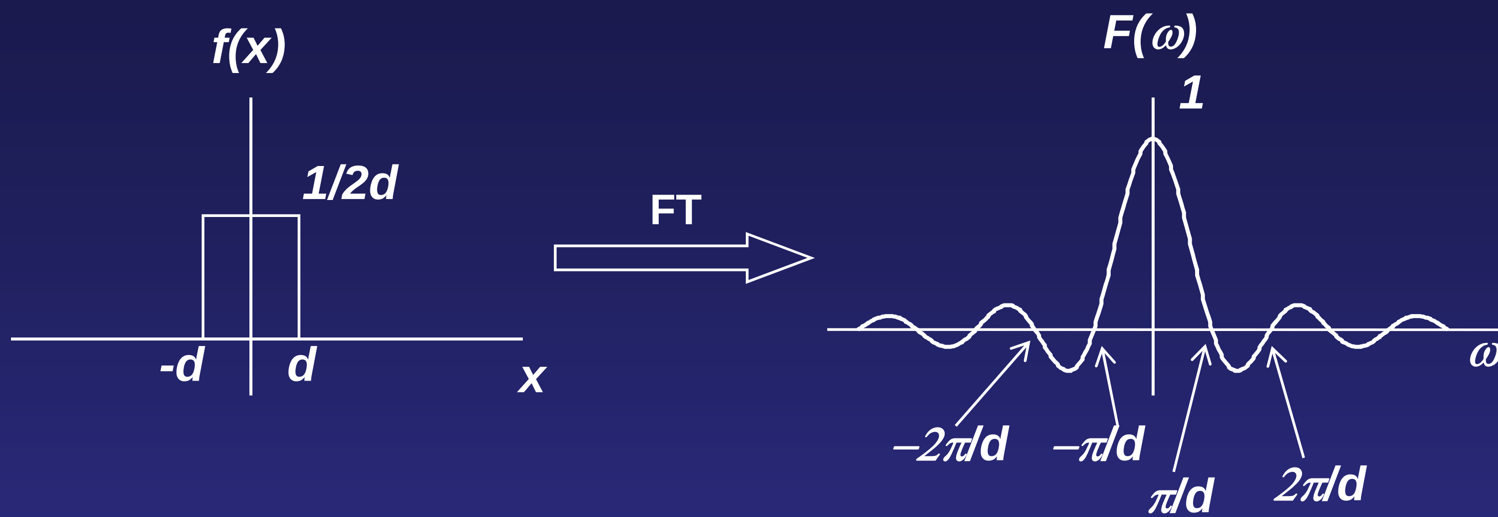
$$= \int_{-d}^{d} \frac{1}{2d} e^{-i\omega x} dx = -\frac{1}{2d}\frac{1}{i\omega}[e^{-i\omega x}]_{-d}^{d}$$

$$= \frac{1}{\omega d} \frac{e^{i\omega d} - e^{-i\omega d}}{2i}$$

$$= \frac{\sin(\omega d)}{\omega d} \qquad (\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i})$$

$$= \sin c(\omega d)$$

Fourier Transform of Rectangle Pulse 2



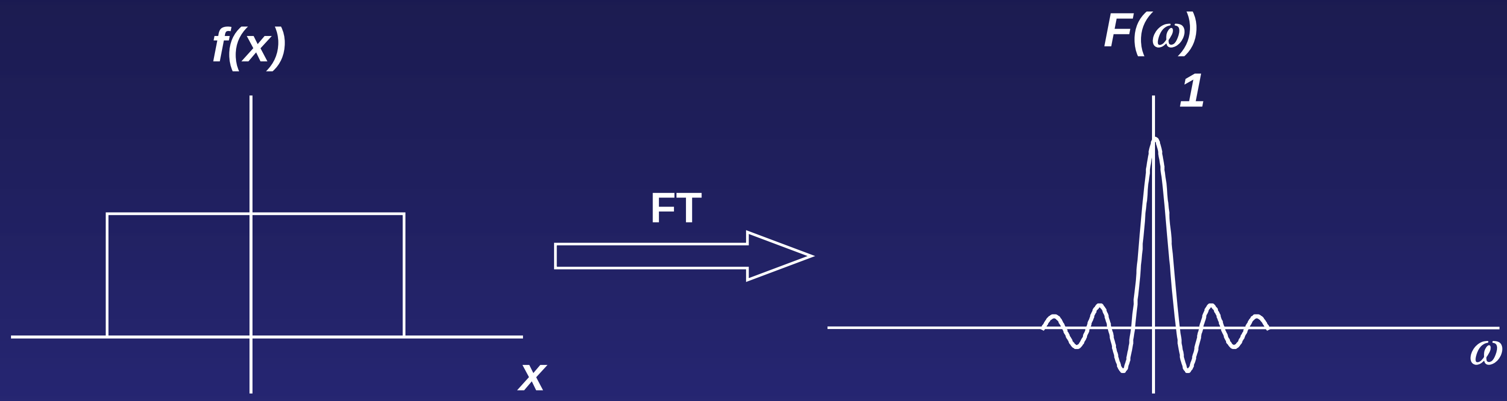
$$f(x)=\begin{cases}1/2d & (|x|\le d)\\ 0 & (|x|>d)\end{cases}$$

$$F(\omega)=\frac{\sin(\omega d)}{\omega d}=\sin c(\omega d)$$

Rectangle Pulse

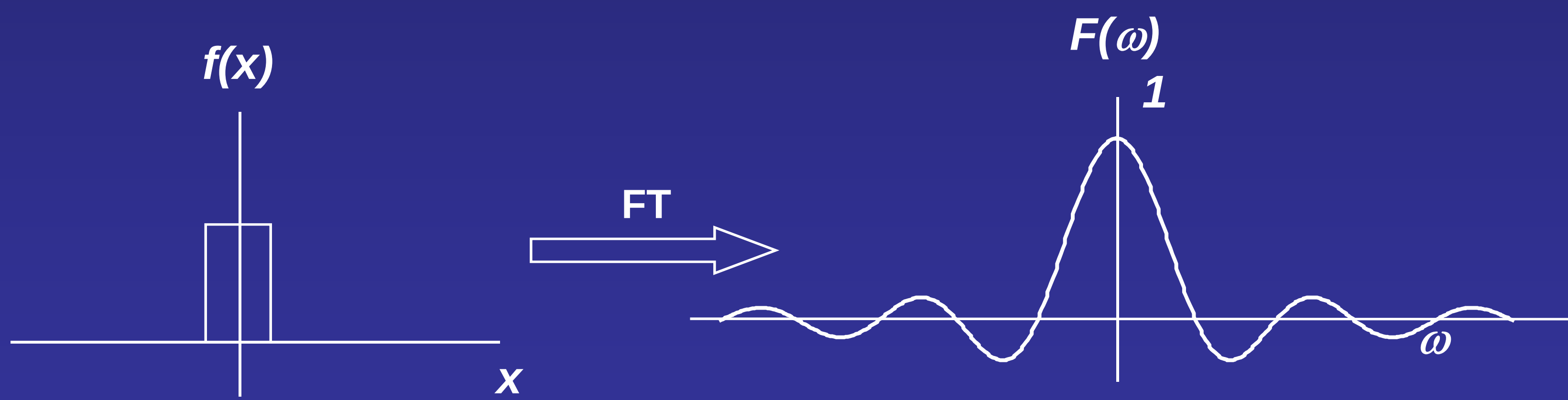
Sinc Function

Relation between Width of Rectangle Pulse and Sinc Function



wide width

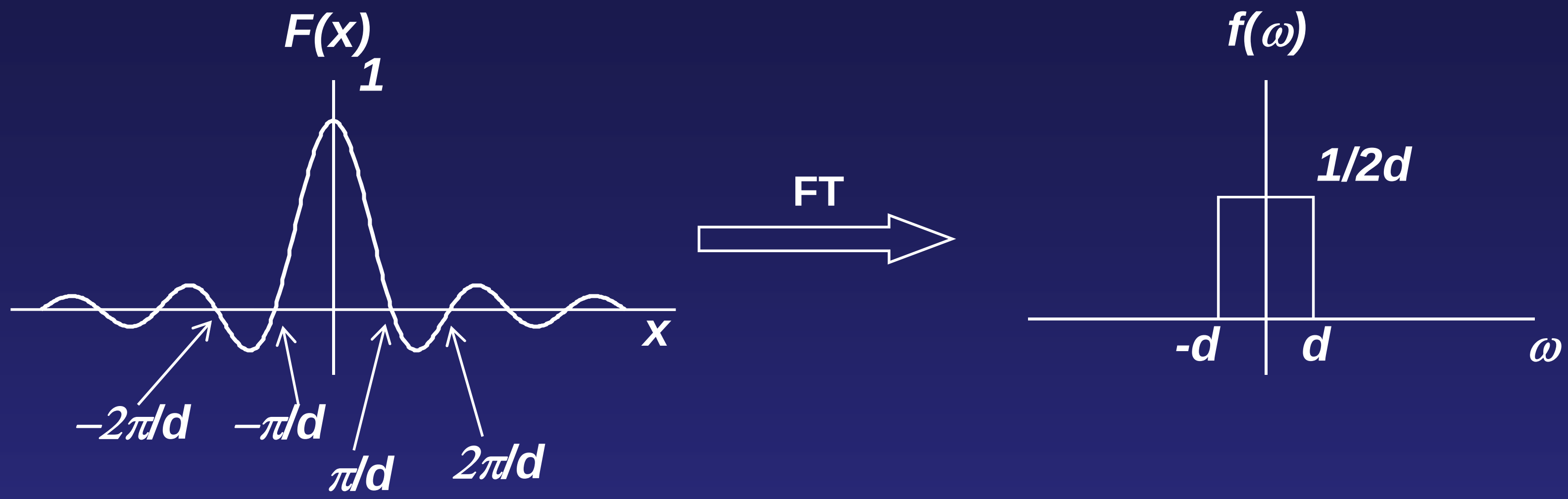
narrow width
(low freq. component)



narrow width

wide width
(high freq. component)

Fourier Transform of Sinc Function



$$F(x) = \frac{\sin(xd)}{xd} = \sin c(xd)$$

$$f(\omega) = \begin{cases} 1/2d & (|\omega| \le d) \\ 0 & (|\omega| > d) \end{cases}$$

Sinc Function

Rectangle Pulse

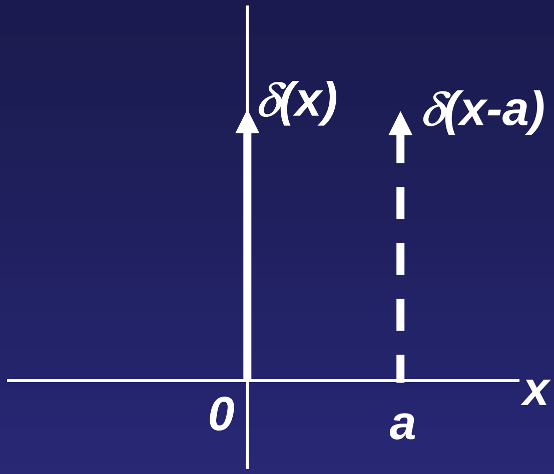
From symmetric property of Fourier transform,

$$if \quad F(\omega) = \Im[f(x)],$$

$$\Im[F(x)] = f(-\omega) = f(\omega) \qquad (f(\omega): even\ function)$$

$$Therefore, \quad \Im\left[\frac{\sin(xd)}{xd}\right] = f(-\omega) = f(\omega) = \begin{cases} 1/2d & (|\omega| \le d) \\ 0 & (|\omega| > d) \end{cases}$$

Fourier Transform of Delta δ-Function 1



Definition of δ-function

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

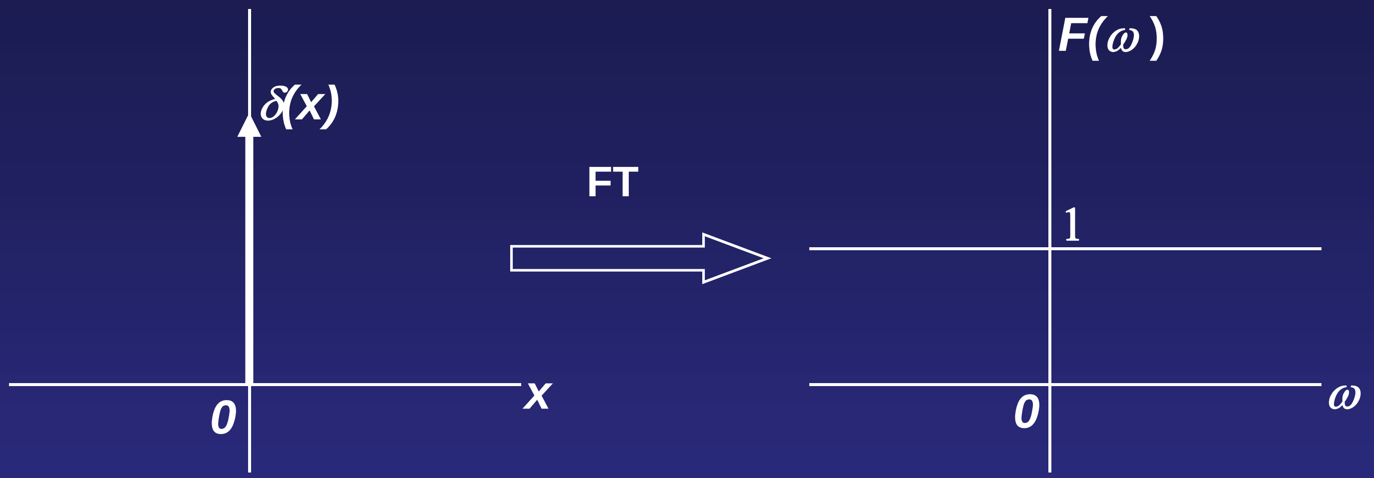
$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

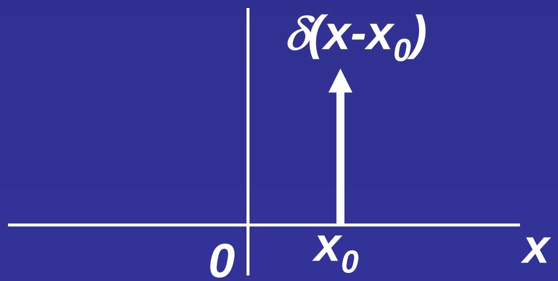
Fourier Transform of δ-function

$$F(\omega) = \int_{-\infty}^{\infty} \delta(x)e^{-i\omega x}dx = e^{-i\omega 0} = e^{0} = 1$$

Fourier Transform of δ-Function 2

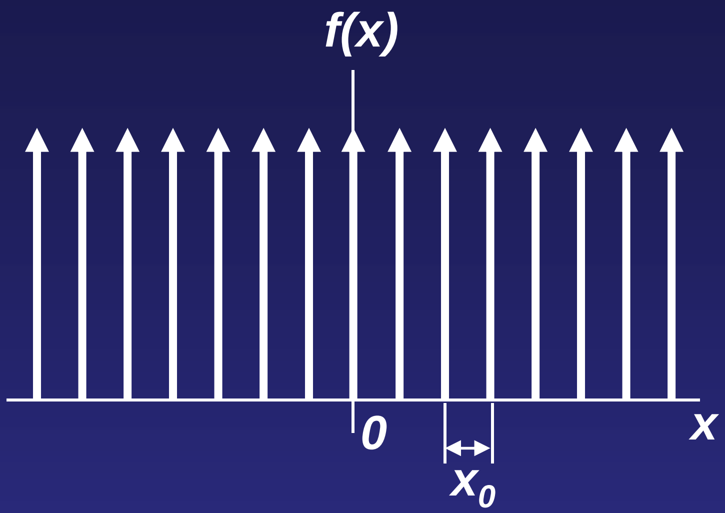


The δ-function includes all frequency component.
The spectrum including all frequency component with constant value is called "white noise".



$$\Im[\delta(x-x_0)] = \int_{-\infty}^{\infty} \delta(x-x_0) e^{-i\omega x} dx = e^{-i\omega x_0}$$

Fourier Transform of δ-Function Sequence 1



δ-function sequence (or comb function):

Periodic function of δ-function with period, x_0

$$f(x) = \cdots + \delta(x+2x_0) + \delta(x+x_0) + \delta(x) + \delta(x-x_0) + +\delta(x-2x_0)\cdots$$

$$= \sum_{n=-\infty}^{\infty} \delta(x - nx_0)$$

Fourier Transform of δ-Function Sequence 2

f(x) can be expressed by a Fourier series, because f(x) is periodic.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n u_0 x} \qquad (\omega_0 = 2\pi u_0)$$

$$where, \quad u_0 = 1/x_0$$

$$c_n = \frac{1}{x_0}\int_{-x_0/2}^{x_0/2} f(x) e^{-i2\pi n u_0 x} dx$$

$$= \frac{1}{x_0}\int_{-x_0/2}^{x_0/2} \delta(x) e^{-i2\pi n u_0 x} dx = \frac{1}{x_0} e^0$$

$$= \frac{1}{x_0} = u_0$$

Therefore,

$$f(x) = \sum_{n=-\infty}^{\infty} u_0 e^{i2\pi n u_0 x} = u_0 \sum_{n=-\infty}^{\infty} e^{i2\pi n u_0 x}$$

Fourier Transform of δ-Function Sequence 3

$$F(u) = \Im[u_0 \sum_{n=-\infty}^{\infty} e^{i2\pi n u_0 x}]$$

$$= u_0 \sum_{n=-\infty}^{\infty} \Im[e^{i2\pi n u_0 x}]$$

$$\Im[\delta(x-u_0)] = e^{-i\omega u_0} = e^{-i2\pi u_0 u}$$

$$\Im[\delta(x+u_0)] = e^{i2\pi u_0 u}$$

$$\Im[\delta(x+nu_0)] = e^{i2\pi n u_0 u}$$

From symmetric property of FT, $\quad if \ F(u) = \Im[f(x)], \quad \Im[F(x)] = f(-u)$

Therefore,

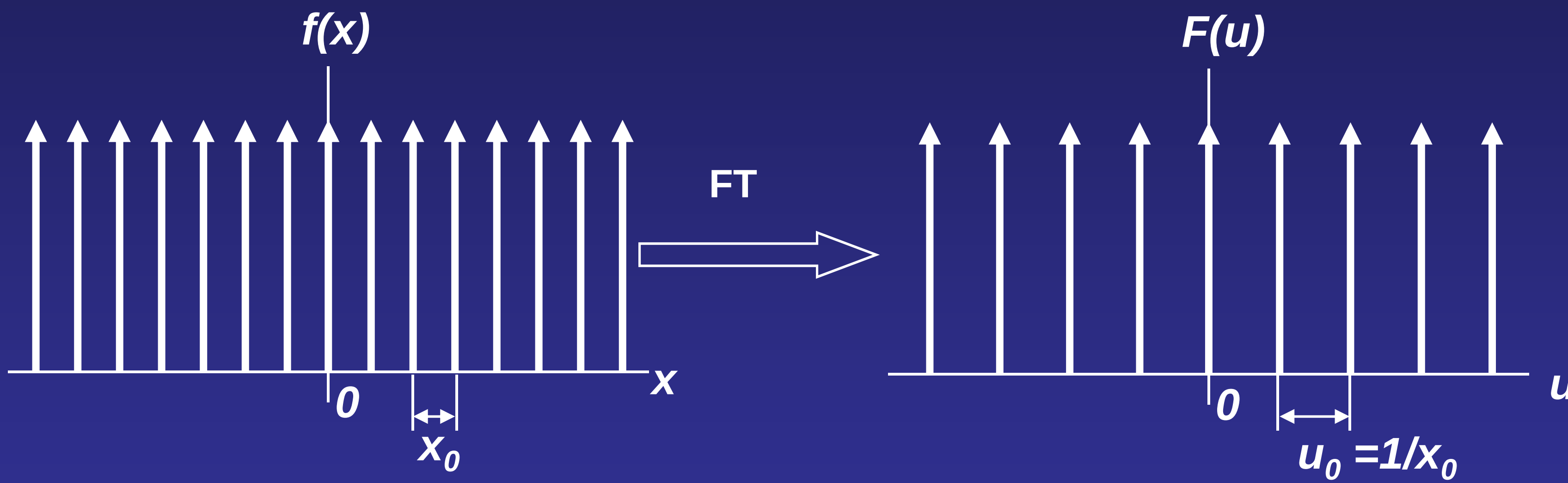
$$\Im[e^{i2\pi n u_0 x}] = \delta(-u+nu_0) = \delta(u-nu_0) \qquad (even\ function: \delta(U) = \delta(-U))$$

Finally,

$$F(u) = u_0 \sum_{n=-\infty}^{\infty} \delta(u-nu_0)$$

Fourier Transform of δ-Function Sequence 4

$$\Im\left[\sum_{n=-\infty}^{\infty}\delta(x-nx_0)\right]=u_0\sum_{n=-\infty}^{\infty}\delta(u-nu_0), \qquad where\, u_0=1/x_0$$



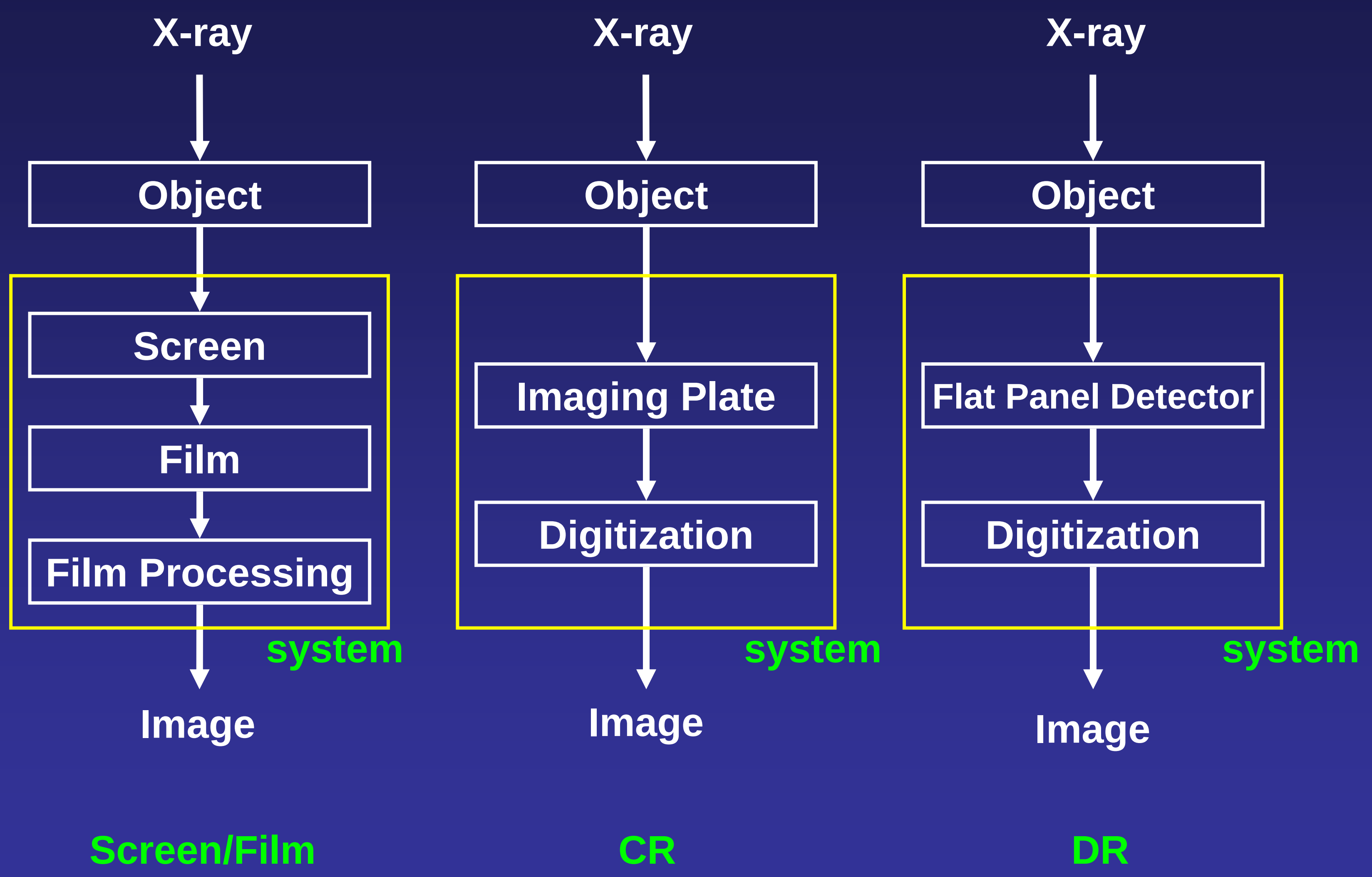
$$f(x)=\sum_{n=-\infty}^{\infty}\delta(x-nx_0)$$

$$F(u)=u_0\sum_{n=-\infty}^{\infty}\delta(u-nu_0)$$

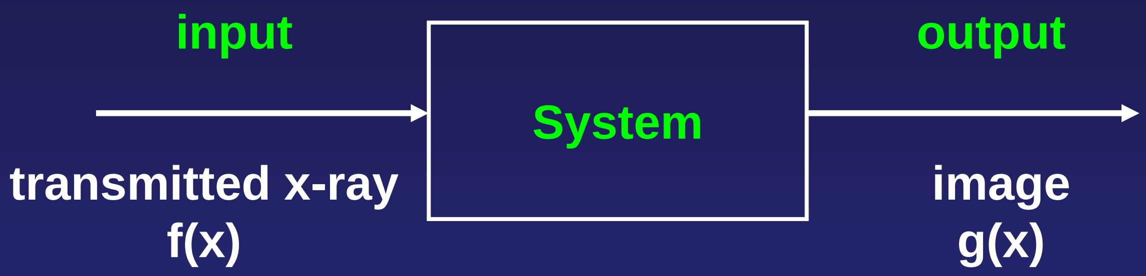
δ-function sequence

δ-function sequence

System for Image Formation



Input-Output System for Image Formation



The system includes screen/film or imaging plate or flat panel detector.

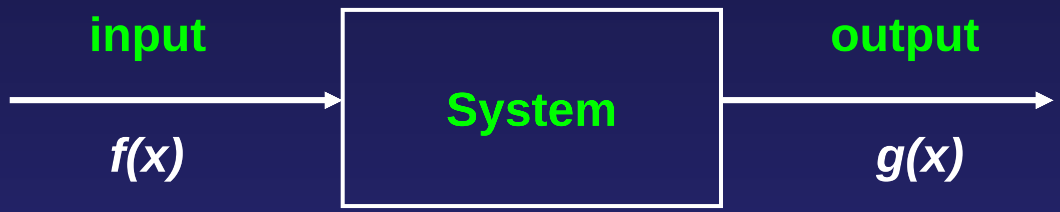
$$g(x) = L[f(x)]$$

where *L* indicate a conversion rule from f(x) to g(x).

If we can know the conversion rule, L, we can extrapolate input from output accurately.

It is very difficult to determine *L* of a general system.
However, *L* of a linear system can be determined easily.

Linear System



Additivity

$$if \;\; g_1(x) = L[f_1(x)], \quad and \;\; g_2(x) = L[f_2(x)],$$

$$L[a_1 f_1(x) + a_2 f_2(x)] = a_1 g_1(x) + a_2 g_2(x)$$

Steadiness

$$L[f(x - x_1)] = g(x - x_1)$$

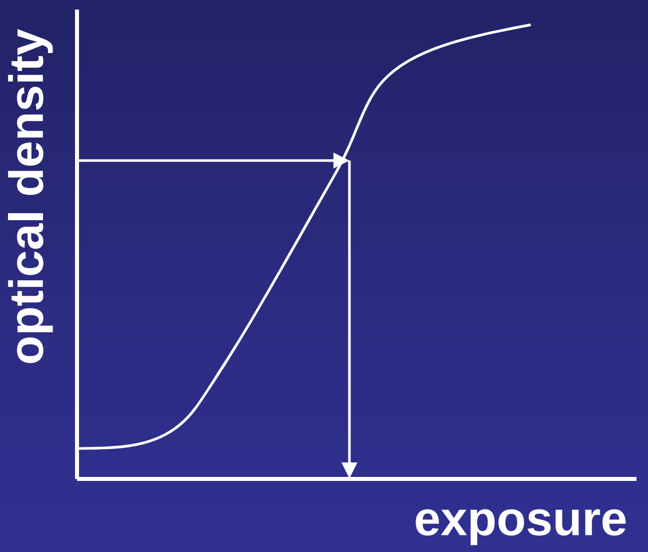
**If the system has additivity and steadiness,
the system is called a linear system.**

Is the screen/film system a linear system?

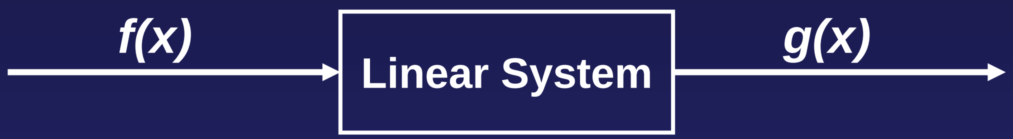
The screen/film system is not linear, because film has not additivity.

However, if the optical density can be converted to exposure by using a film characteristic curve, we can handle the screen/film system as a linear system.

Film Characteristic Curve



Linear System Response 1

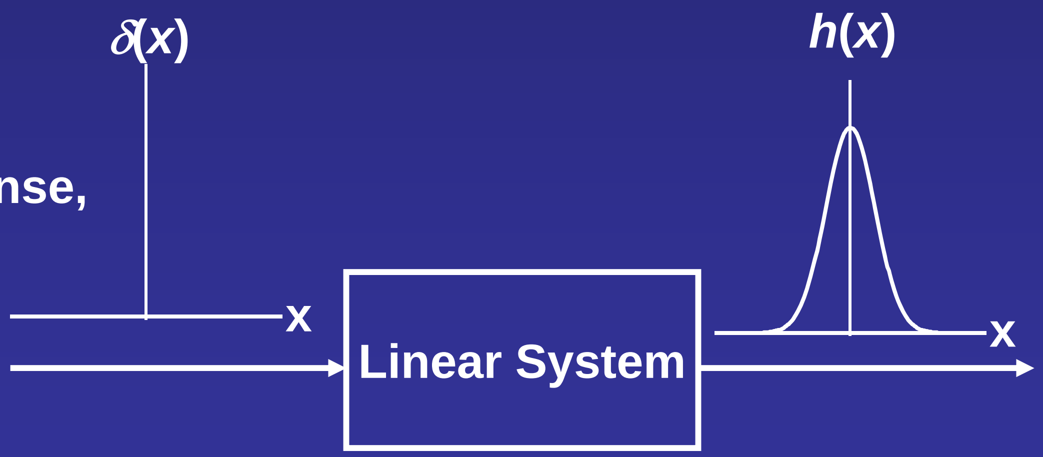


By using δ-function, input *f(x)* can be expressed as follows,

$$f(x) = \int_{-\infty}^{\infty} f(\tau)\delta(x-\tau)d\tau \qquad because \int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

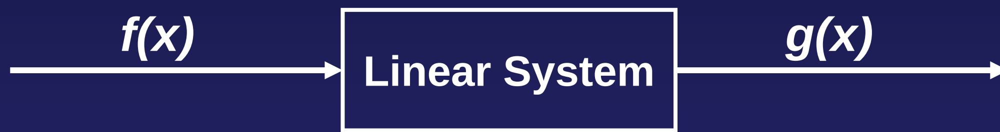
Consider the impulse response,

$$h(x) = L[\delta(x)]$$



Impulse Response

Linear System Response 2



$$
\begin{aligned}
g(x) &= L[f(x)] \\
&= L\left[\int_{-\infty}^{\infty} f(\tau)\delta(x-\tau)\,d\tau\right] \\
&= \int_{-\infty}^{\infty} f(\tau)L[\delta(x-\tau)]\,d\tau \\
&= \int_{-\infty}^{\infty} f(\tau)h(x-\tau)\,d\tau \\
&= f(x) * h(x)
\end{aligned}
$$

Linear System Response 3

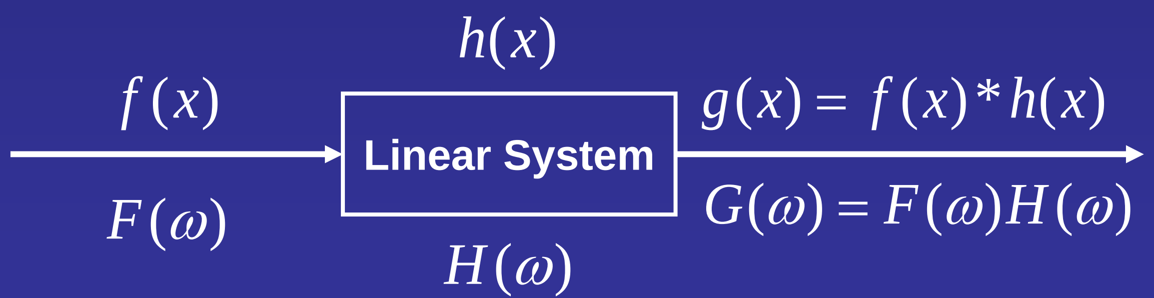
In summary, output of a linear system can be obtained from convolution integral of input and impulse response as follows,

$$g(x) = f(x) * h(x)$$

$$\text{if} \;\; F(\omega) = \Im[f(x)], \quad G(\omega) = \Im[g(x)], \text{ and } H(\omega) = \Im[h(x)],$$

from convolution integral theorem,

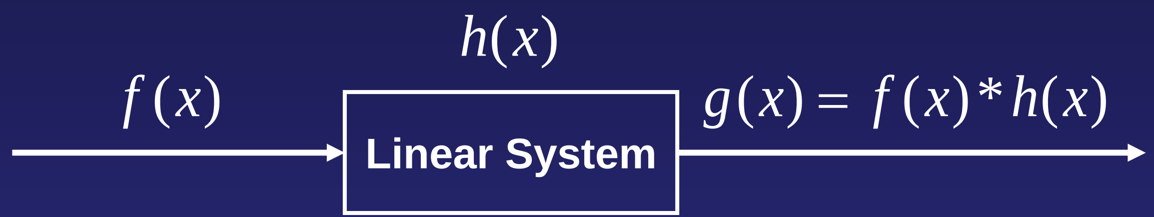
$$G(\omega) = F(\omega)H(\omega)$$

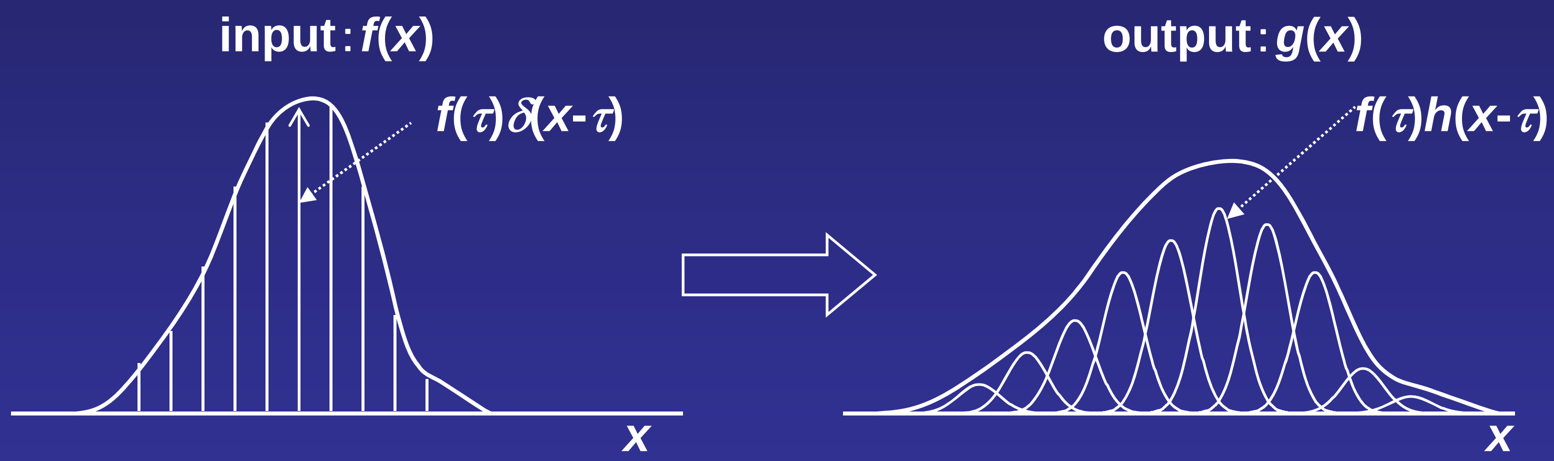


:in real domain

:in frequency domain

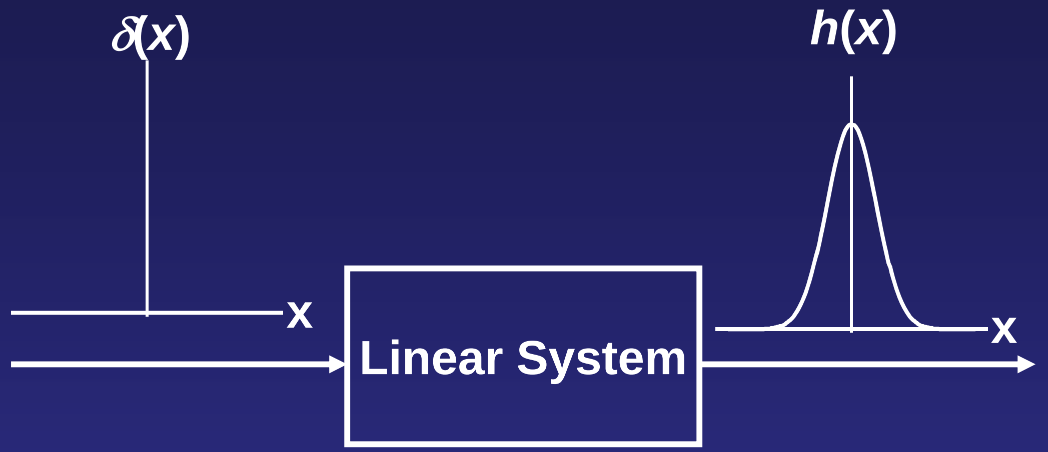
Linear System Response in Real Domain





The output of a linear system is superposition of many impulse responses.

System Function (Frequency Response Function) 1



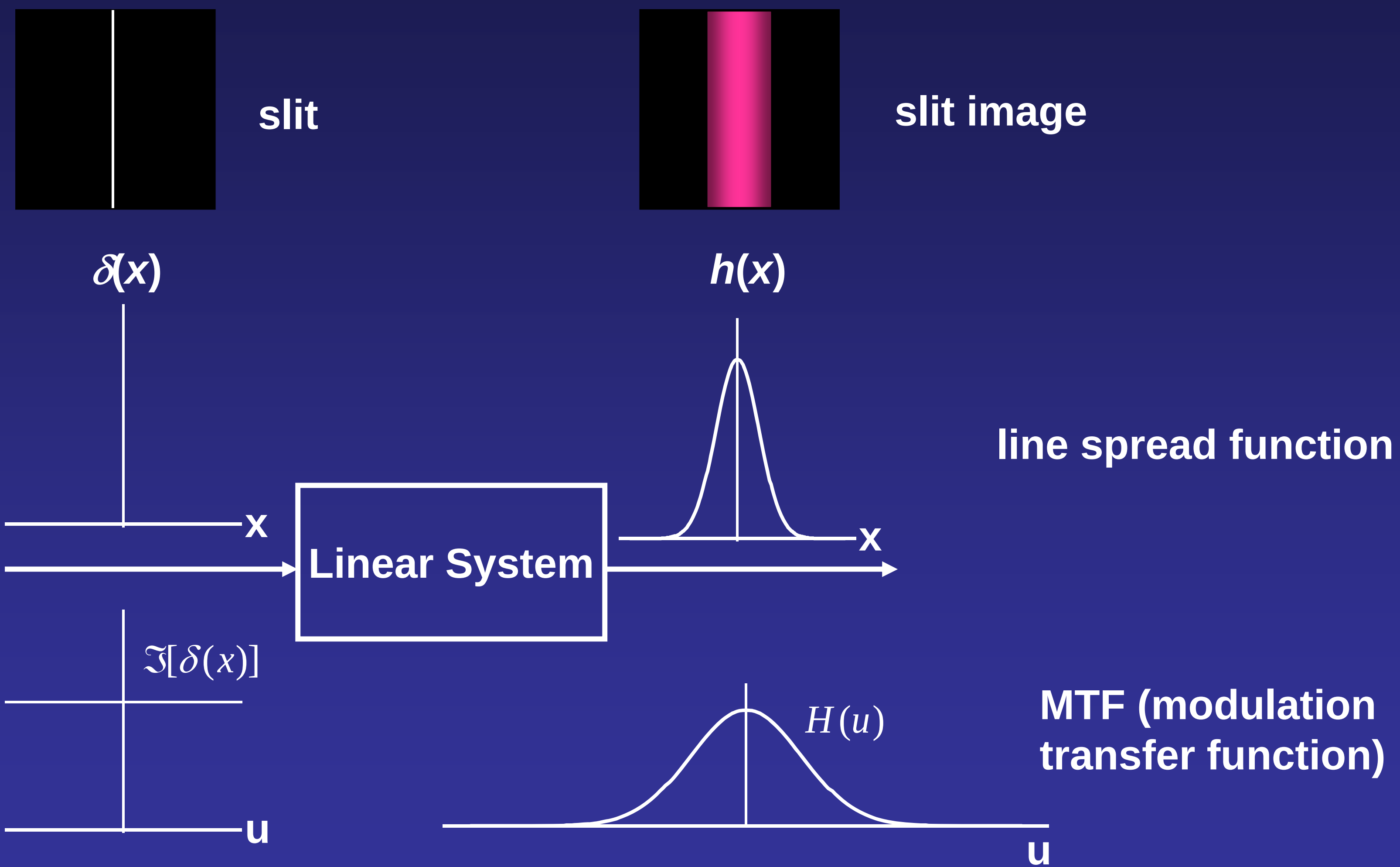
h(x): **impulse response**

The Fourier transform of impulse response, *h(x),* is called the system function, $H(\omega)$.

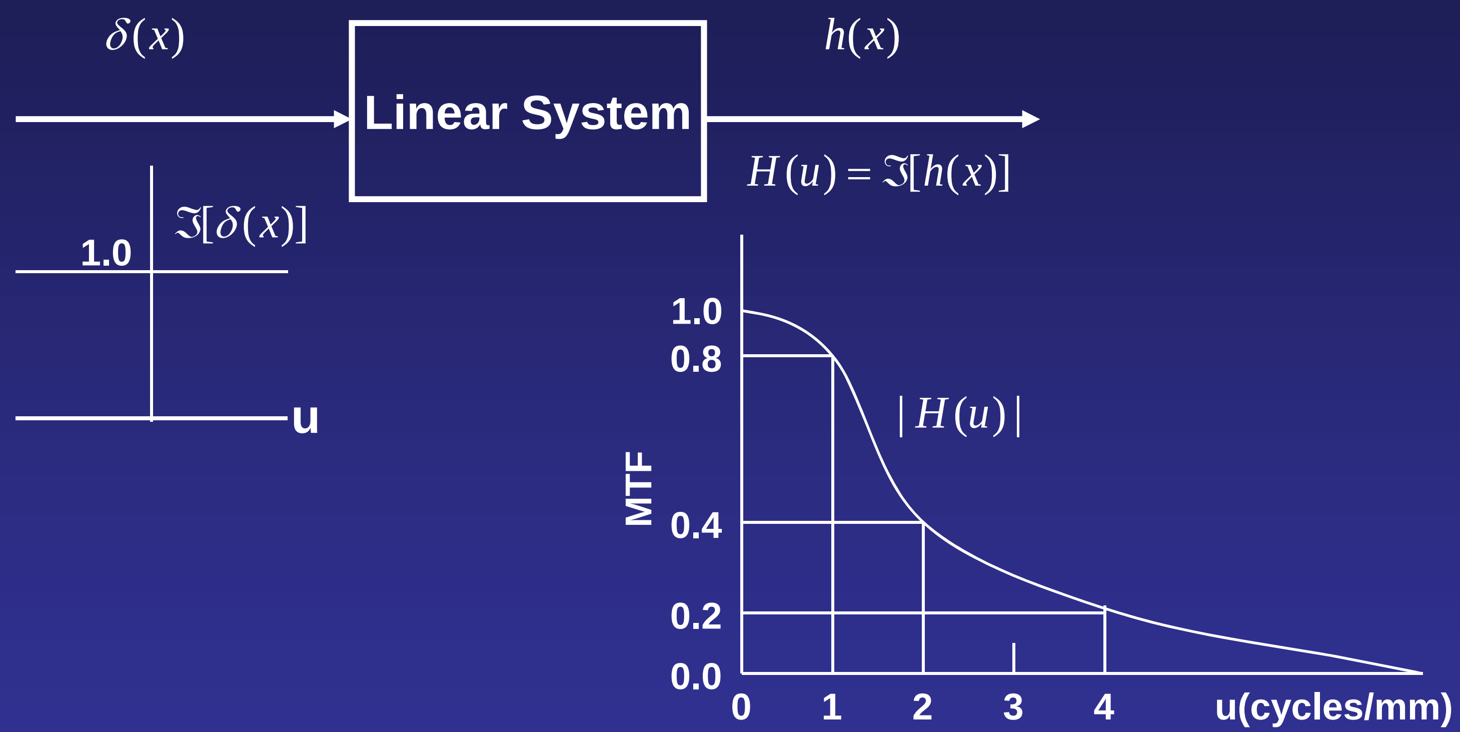
The impulse response of screen/film is obtained by a slit image (line spread function) with very narrow width.

$|H(\omega)|$ is called MTF or the response function to evaluate the frequency response of contrast on medical images .

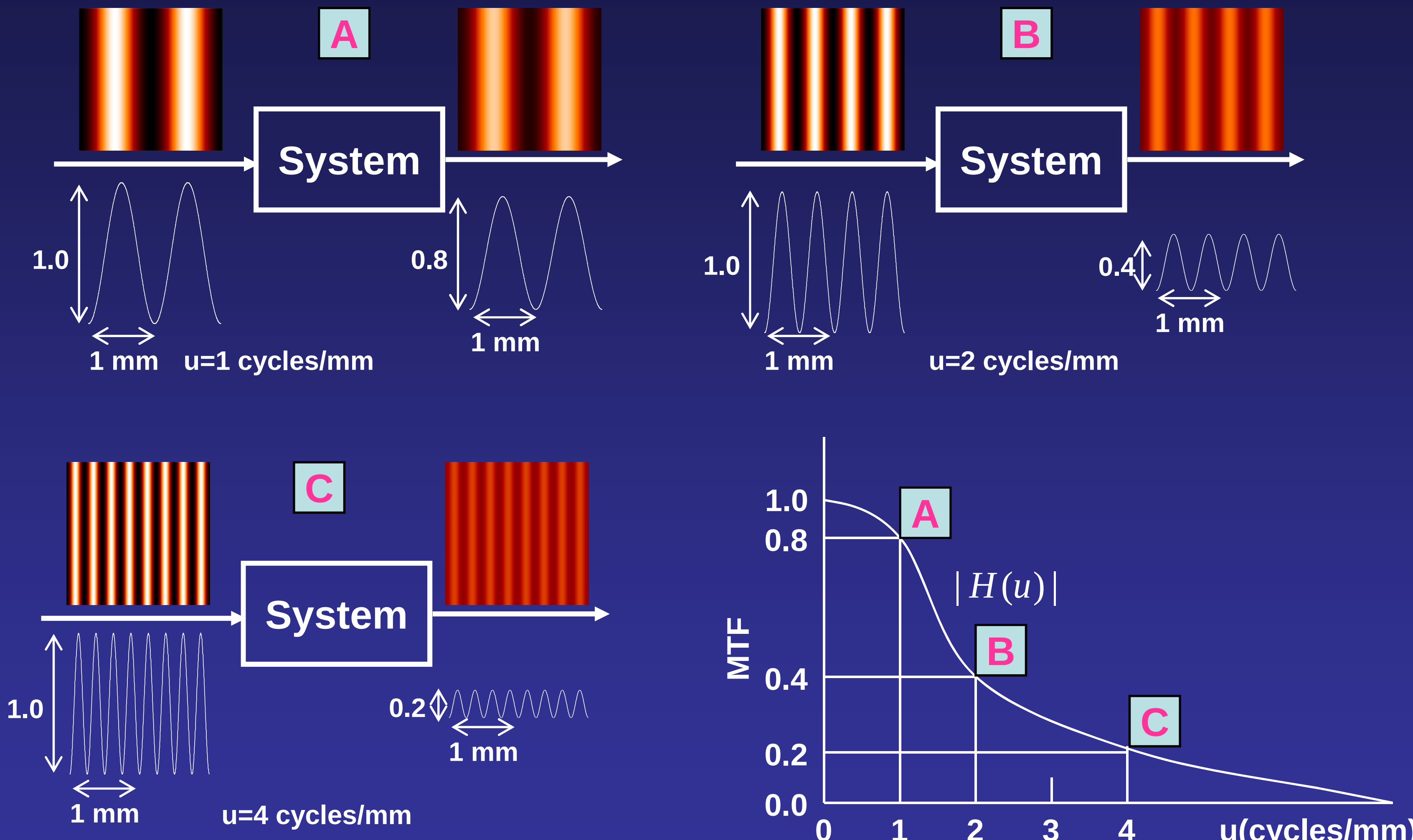
Impulse Response of Screen/Film



Modulation Transfer Function (MTF) 1



Modulation Transfer Function (MTF) 2



Two Dimensional (2D) Fourier Transform

1D FT $F(u)=\int_{-\infty}^{\infty} f(x)e^{-i2\pi ux}dx,$ $f(x)=\int_{-\infty}^{\infty} F(u)e^{i2\pi ux}du$

2D FT can be obtained from the expansion of 1D FT.

$$F(u,v)=\Im[f(x,y)]$$

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(ux+vy)}dxdy$$

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{i2\pi(ux+vy)}dudv$$

Two Dimensional (2D) Discrete Fourier Transform (DFT)

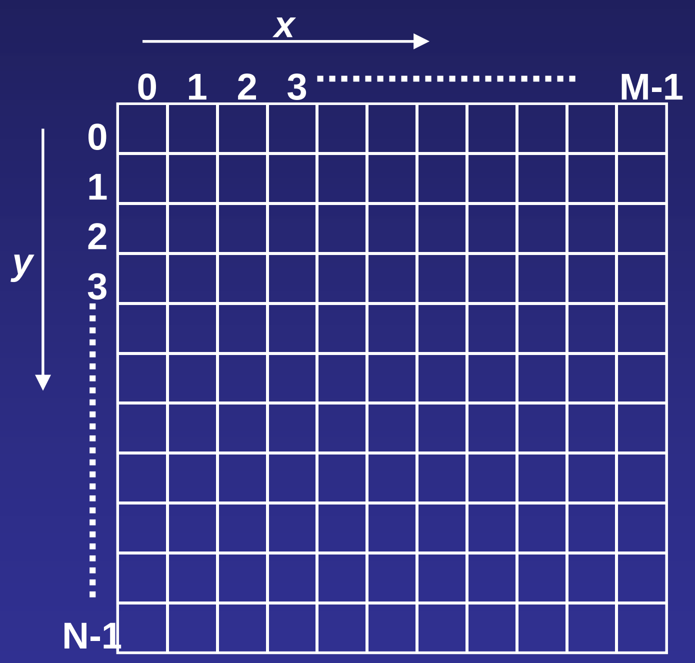
2D DFT of a digital image with M x N matrix size

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-i2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{i2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$x, u = 0, 1, 2, \cdots\cdots, M-1$$

$$y, v = 0, 1, 2, \cdots\cdots, N-1$$



Digital Image

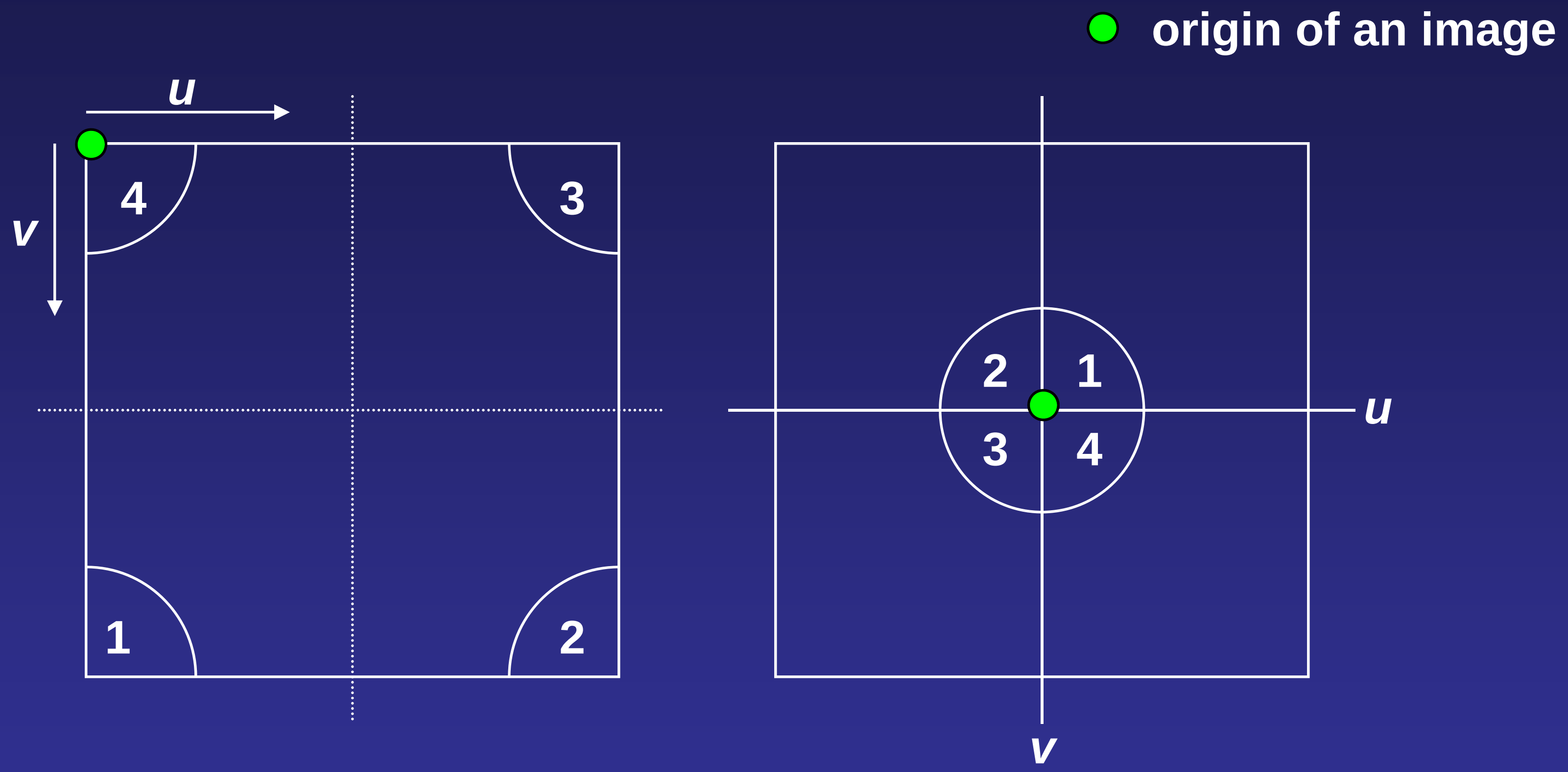
$$\because e^{i2\pi k} = 1 \quad (k = \pm 1, \pm 2, \pm 3, \cdots)$$

$$f(x + k_x M, y + k_y N) = f(x,y)$$

$$k_x, k_y = \pm 1, \pm 2, \pm 3, \cdots$$

FFT (fast Fourier transform) is usually used for calculation.

Rearrangement Frequency Domain after FT



FFT is calculated for image with an origin at upper left. Therefore, we must rearrange the frequency domain to move an origin from upper left to center.

Thank you for your attention!!